Extended Appendix to
"Acquiring Information Through Peers"

Bernard Herskovic* João Ramos†
UCLA Anderson USC Marshall

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This is an extended appendix to our paper "Acquiring Information Through Peers", containing detailed derivations omitted in the paper. In Appendix A.1 and A.2, we prove Propositions 1 and 2 for a more general version of the model with conformity and signals' precision being agent-specific, i.e. $\{r_i, \sigma_i\}_{i=1}^n$. In Appendix B.1, we prove Proposition 3, and, in B.2, we show an example in which Properties 1 and 2 do not hold in an out-of-equilibrium network. In AppendixB.3, we prove Proposition 4, and, in Appendix B.4, we show an example of an equilibrium with a three-tier network. In Appendix C, we prove Theorem 3. Finally, we prove the Propositions 5 and 6 in Appendix D.

A Proof of Propositions 1 and 2

We prove Proposition 1 in Section A.1 and Proposition 2 in Section A.2.

A.1 Proof of Proposition 1

We prove Proposition 1 in two steps. First, we fully characterize the linear equilibrium in Section A.1.1. Second, we show that there is a unique equilibrium in the first stage of the game in Section A.1.3.

A.1.1 Linear Equilibrium

Let $\overline{a} \equiv \frac{1}{n}\sum_{j=1}^n a_j$ be the average action, and let $\overline{a}_{-i} \equiv \frac{1}{n-1}\sum_{j\neq i} a_j = \frac{n}{n-1}\overline{a} - \frac{1}{n-1}a_i$ be the average action without agent i. We will verify the following guess:

$$\overline{a} = \sum_{j=0}^{n} \beta_j e_j \tag{DD.1}$$

From the first order condition, agent i's optimal action satisfies:

*Email: bernard.herskovic@anderson.ucla.edu.
†Email: joao.ramos@marshall.usc.edu.

$$
\begin{aligned}
a_i &= (1-r_i)\mathbb{E}\left[\theta|\mathbb{I}_i\right] + r_i E\left[\overline{a}_{-i}|\mathbb{I}_i\right] \\
&= (1-r_i)\mathbb{E}\left[\theta|\mathbb{I}_i\right] + r_i E\left[\tfrac{n}{n-1}\overline{a} - \tfrac{1}{n-1}a_i|\mathbb{I}_i\right] \\
&= (1-\tilde{r}_i)\mathbb{E}\left[\theta|\mathbb{I}_i\right] + \tilde{r}_i E\left[\overline{a}|\mathbb{I}_i\right]
\end{aligned}
$$

where $\tilde{r}_i = \frac{r_i n}{r_i+n-1}$. Using bayes updating, the expected value of the state of the world given agent i's informational set is given by $\mathbb{E}\left[\theta|\mathbb{I}_i\right] = \sum_{j=0}^{n} \tilde{g}_{ij} e_j \equiv \overline{e}_i$, where $\tilde{g}_{ij} = \frac{g_{ij}\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}$, $e_0 = 0$ is the prior's mean and $\sigma_0 = 1$ is the prior standard deviation. The expected value of the average action given i's information is given by

$$
E\left[\overline{a}|\mathbb{I}_i\right] = \sum_{j=0}^{n} \beta_j E\left[e_j|\mathbb{I}_i\right] = \sum_{j=0}^{n} \beta_j g_{ij} e_j + \sum_{j=0}^{n} \beta_j (1-g_{ij})\overline{e}_i
$$

Thus, player i's action is simplified to

$$
a_i = (1-\tilde{r}_i)\overline{e}_i + \tilde{r}_i \sum_{j=0}^{n} \beta_j g_{ij} e_j + \tilde{r}_i \sum_{j=0}^{n} \beta_j (1-g_{ij})\overline{e}_i
$$

In order to verify our initial guess, we take the average over i,

$$
\begin{aligned}
\overline{a} &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n} a_i \\
n\overline{a} &= \textstyle\sum_{i=1}^{n}\left[(1-\tilde{r}_i)\overline{e}_i + \tilde{r}_i \sum_{j=0}^{n}\beta_j g_{ij} e_j + \tilde{r}_i \sum_{j=0}^{n}\beta_j (1-g_{ij})\overline{e}_i\right] \\
&= \textstyle\sum_{i=1}^{n} \overline{e}_i - \sum_{i=1}^{n} \tilde{r}_i \overline{e}_i + \sum_{j=0}^{n}\beta_j \sum_{i=1}^{n} \tilde{r}_i g_{ij} e_j + \sum_{j=0}^{n}\beta_j \sum_{i=1}^{n} \tilde{r}_i \overline{e}_i - \sum_{j=0}^{n}\beta_j \sum_{i=1}^{n} \tilde{r}_i g_{ij} \overline{e}_i
\end{aligned}
$$

Using matrix notation let

$$
\tilde{r} = \begin{bmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_n \end{bmatrix}_{n\times 1}, \quad \overline{e} = \begin{bmatrix} \overline{e}_1 \\ \vdots \\ \overline{e}_n \end{bmatrix}_{n\times 1}, \quad e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}_{n\times 1}, \quad G = \begin{bmatrix} g_{10} & g_{11} & \cdots & g_{1n} \\ \vdots & & \ddots & \\ g_{n0} & g_{n1} & \cdots & g_{nn} \end{bmatrix}_{n\times n+1}
$$

Hence, the sum of all action becomes

$$
n\overline{a} = \mathbf{1}'\overline{e} - \tilde{r}'\overline{e} + \beta' \mathrm{diag}(G'\tilde{r}) \begin{bmatrix} 0 \\ e \end{bmatrix} + \beta' \mathbf{1} \tilde{r}' \overline{e} - \beta' G' \mathrm{diag}(\tilde{r})\overline{e}
$$

where $\mathbf{1}$ is a column vector of ones with the appropriate dimension, and diag$(\cdot)$ creates a diagonal matrix. Let

$$
\tilde{G} = \begin{bmatrix} \tilde{g}_{10} & \tilde{g}_{11} & \cdots & \tilde{g}_{1n} \\ \vdots & & \ddots & \\ \tilde{g}_{n0} & \tilde{g}_{n1} & \cdots & \tilde{g}_{nn} \end{bmatrix}_{n\times n+1}
$$

and we have that $\overline{e} = \tilde{G}\begin{bmatrix} 0 \\ e \end{bmatrix}$. The sum of actions becomes

$$\begin{aligned} n\overline{a} &= \mathbf{1}'\tilde{G}\begin{bmatrix} 0 \\ e \end{bmatrix} - \tilde{r}'\tilde{G}\begin{bmatrix} 0 \\ e \end{bmatrix} + \beta'\text{diag}(G'\tilde{r})\begin{bmatrix} 0 \\ e \end{bmatrix} + \beta'\mathbf{1}\tilde{r}'\tilde{G}\begin{bmatrix} 0 \\ e \end{bmatrix} - \beta'G'\text{diag}(\tilde{r})\tilde{G}\begin{bmatrix} 0 \\ e \end{bmatrix} \\ &= (\mathbf{1}-\tilde{r})'\,\tilde{G}\begin{bmatrix} 0 \\ e \end{bmatrix} + \beta'\left(\text{diag}(G'\tilde{r}) + \mathbf{1}\tilde{r}'\tilde{G} - G'\text{diag}(\tilde{r})\tilde{G}\right)\begin{bmatrix} 0 \\ e \end{bmatrix} \\ \overline{a} &= \tfrac{1}{n}\left\{(\mathbf{1}-\tilde{r})'\,\tilde{G} + \beta'\left(\text{diag}(G'\tilde{r}) + \mathbf{1}\tilde{r}'\tilde{G} - G'\text{diag}(\tilde{r})\tilde{G}\right)\right\}\begin{bmatrix} 0 \\ e \end{bmatrix} \end{aligned}$$

Next, we use method of undetermined coefficient to solve for the vector of loadings β based on our initial guess $\overline{a} = \beta'\begin{bmatrix} 0 \\ e \end{bmatrix}$.

$$\begin{aligned} \beta' &= \frac{1}{n}\left\{(\mathbf{1}-\tilde{r})'\,\tilde{G} + \beta'\left(\text{diag}(G'\tilde{r}) + \mathbf{1}\tilde{r}'\tilde{G} - G'\text{diag}(\tilde{r})\tilde{G}\right)\right\} \qquad \text{(DD.2)} \\ \beta' &= \frac{1}{n}(\mathbf{1}-\tilde{r})'\,\tilde{G}\left[\mathrm{I} - \frac{1}{n}\left(\text{diag}(G'\tilde{r}) + \mathbf{1}\tilde{r}'\tilde{G} - G'\text{diag}(\tilde{r})\tilde{G}\right)\right]^{-1} \end{aligned}$$

We can verify that the average action loadings sum to 1. Starting from the equation above and post-multiplying by a vector of ones.

$$\begin{aligned} n\beta'\mathbf{1} &= (\mathbf{1}-\tilde{r})'\,\tilde{G}\mathbf{1} + \beta'\left(\text{diag}(G'\tilde{r})\mathbf{1} + \mathbf{1}\tilde{r}'\tilde{G}\mathbf{1} - G'\text{diag}(\tilde{r})\tilde{G}\mathbf{1}\right) \\ n\beta'\mathbf{1} &= (\mathbf{1}-\tilde{r})'\,\mathbf{1} + \beta'\left(G'\tilde{r} + \mathbf{1}\tilde{r}'\mathbf{1} - G'\text{diag}(\tilde{r})\mathbf{1}\right) \\ n\beta'\mathbf{1} &= n - \tilde{r}'\mathbf{1} + \beta'G'\tilde{r} + \beta'\mathbf{1}\tilde{r}'\mathbf{1} - \beta'G'\tilde{r} \\ \beta'\mathbf{1}(n - \tilde{r}'\mathbf{1}) &= n - \tilde{r}'\mathbf{1} \\ \beta'\mathbf{1} &= 1 \qquad \text{(DD.3)} \end{aligned}$$

The action of each agent in vector notation is given by

$$\begin{aligned} a &= \overline{e} - \text{diag}(\tilde{r})\overline{e} + \text{diag}(\tilde{r})G\text{diag}(\beta)\begin{bmatrix} 0 \\ e \end{bmatrix} + \text{diag}(\tilde{r})\overline{e} - \text{diag}(\beta'G')\text{diag}(\tilde{r})\overline{e} \\ &= \left[\tilde{G} - \text{diag}(\tilde{r})\tilde{G} + \text{diag}(\tilde{r})G\text{diag}(\beta) + \text{diag}(\tilde{r})\tilde{G} - \text{diag}(\beta'G')\text{diag}(\tilde{r})\tilde{G}\right]\begin{bmatrix} 0 \\ e \end{bmatrix} \\ &= \Lambda\begin{bmatrix} 0 \\ e \end{bmatrix} \end{aligned}$$

where Λ is a $n \times n + 1$ matrix of loadings

$$\Lambda = \tilde{G} - \text{diag}(\tilde{r})\tilde{G} + \text{diag}(\tilde{r})G\text{diag}(\beta) + \text{diag}(\tilde{r})\tilde{G} - \text{diag}(\beta'G')\text{diag}(\tilde{r})\tilde{G}, \qquad \text{(DD.4)}$$

A.1.2 Solving for λ_{ij} and β_j using sum notation

Let λ_{ij} be the element (i, j) in the matrix Λ. Following Equation (DD.4), the λs in sum notation are given by:

$$\lambda_{ij} = (1 - \tilde{r}_i)\,\tilde{g}_{ij} + \tilde{r}_i\beta_j g_{ij} + \tilde{r}_i\tilde{g}_{ij} - \tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\tilde{g}_{ij} \tag{DD.5}$$

for $i = 1, \ldots, n$, and $j = 0, \ldots, n$. Notice that $\sum_{j=0}^{n}\lambda_{ij} = 1$, since $\sum_{j=0}^{n}\tilde{g}_{ij} = 1$. Also notice that $\lambda_{ij} = 0$ whenever $g_{ij} = 0$, and $\lambda_{ij} > 0$ whenever $g_{ij} = 1$.

If $r_i = r$ for every $i = 1, \ldots, n$ and $\sigma_j = \sigma$ for every $j = 1, \ldots, n$, then

$$\lambda_{ij} = (1 - \tilde{r})\,\frac{g_{ij}}{\mathcal{K}_i + 1 + \sigma^2} + \tilde{r}\beta_j g_{ij} + \tilde{r}_i\frac{g_{ij}}{\mathcal{K}_i + 1 + \sigma^2} - \tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{g_{ij}}{\mathcal{K}_i + 1 + \sigma^2} \tag{DD.6}$$

Next, we use Equation (DD.2) to derive βs in sum notation:

$$\begin{aligned} n\beta' &= (\mathbf{1} - \tilde{r})'\,\tilde{G} + \beta'\mathrm{diag}(G'\tilde{r}) + \beta'\mathbf{1}\tilde{r}'\tilde{G} - \beta' G'\mathrm{diag}(\tilde{r})\tilde{G} \\ n\beta_j &= \textstyle\sum_{i=1}^{n}(1 - \tilde{r}_i)\tilde{g}_{ij} + \beta_j\sum_{i=1}^{n}\tilde{r}_i g_{ij} + \sum_{i=1}^{n}\tilde{r}_i\tilde{g}_{ij} - \sum_{i=1}^{n}\sum_{s=0}^{n}\beta_s g_{is}\tilde{r}_i\tilde{g}_{ij} \end{aligned}$$

Given that $\sum_{j=0}^{n}\beta_j = 1$ (Equation DD.3), we have:

$$n\beta_j = \sum_{i=1}^{n}\tilde{g}_{ij} + \beta_j\sum_{i=1}^{n}\tilde{r}_i g_{ij} - \sum_{i=1}^{n}\sum_{s=0}^{n}\beta_s g_{is}\tilde{r}_i\tilde{g}_{ij} \tag{DD.7}$$

Also, notice that by definition we have:

$$\beta_j = \frac{1}{n}\sum_{i=1}^{n}\lambda_{ij}. \tag{DD.8}$$

If $r_i = r$ for every $i = 1, \ldots, n$ and $\sigma_j = \sigma$ for every $j = 1, \ldots, n$, then

$$\beta_j = \frac{1}{n}\sum_{i=1}^{n}\frac{g_{ij}}{\sigma^2 + \mathcal{K}_i + 1} + \frac{\tilde{r}}{n}\left[\beta_j\left(\overline{\mathcal{K}}_j + 1\right) - \sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is} g_{ij}}{\sigma^2 + \mathcal{K}_i + 1}\right] \tag{DD.9}$$

where

$$\overline{\mathcal{K}}_j = \sum_{s=1, s\neq j}^{n} g_{sj},$$

and

$$\mathcal{K}_i = \sum_{s=1, s\neq i}^{n} g_{is}.$$

Since $\sum_{s=0}^{n}\beta_s = 1$, we can rearrange the expression above as follows:

$$\left[n - \tilde{r}\left(\overline{\mathcal{K}}_j + 1\right)\right]\beta_j = (1 - \tilde{r})\sum_{i=1}^{n}\frac{g_{ij}}{\sigma^2 + \mathcal{K}_i + 1} + \tilde{r}\sum_{i=1}^{n}\frac{g_{ij}}{\mathcal{K}_i + \sigma^2 + 1}\left[\sum_{s=0}^{n}\beta_s\,(1 - g_{is})\right],$$

which is Equation 2 in the main text.

If $\sigma = 1$, then we have

$$\beta_j = \frac{1}{n}\sum_{i=1}^{n}\frac{g_{ij}}{\mathcal{K}_i + 2} + \frac{\tilde{r}}{n}\left[\beta_j\left(\overline{\mathcal{K}}_j + 1\right) - \sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is} g_{ij}}{\mathcal{K}_i + 2}\right]$$

Finally, the loading $\beta_{-i,j}$ is by definition given by

$$\beta_{-i,j} = \frac{1}{n-1}\sum_{k\neq i}\lambda_{ij}, \tag{DD.10}$$

which can be written as:

$$\beta_{-i,j} = \frac{n}{n-1}\beta_j - \frac{1}{n-1}\lambda_{ij}. \tag{DD.11}$$

Notice that

$$\sum_{j=0}^{n}\beta_{-i,j} = 1 \tag{DD.12}$$

because $\sum_{j=0}^{n}\beta_j = 1$ and $\sum_{j=0}^{n}\lambda_{i,j} = 1$, as shown earlier.

A.1.3 Uniqueness

In this section, we prove that there is a unique equilibrium in the second stage of the game. We follow closely the uniqueness proof in Hellwig and Veldkamp (2009), but adapted to our setting. We focus on symmetric equilibria, that is, agents act in an identical manner if they have the same information set. We also assume $r_i = r$ for every $i = 1, \ldots, n$. From the first-order condition, in any equilibrium agent i's optimal action satisfies:

$$a_i = (1-\tilde{r})\mathbb{E}\left[\theta|\mathbb{I}_i\right] + \tilde{r}E\left[\overline{a}|\mathbb{I}_i\right]$$

To prove uniqueness, we procedure in two steps. First, we show that the following expression constitutes the unique solution to agents' first-order conditions:

$$a_i = a(\mathbb{I}_i) = (1-\tilde{r})\sum_{t=1}^{\infty}\tilde{r}^t\mathbb{E}_i\left[\overline{\mathbb{E}}^t(\theta)\right] \tag{DD.13}$$

where $\mathbb{E}_i(\cdot) = \mathbb{E}_i[\cdot|\mathbb{I}_i]$, $\overline{\mathbb{E}}(\cdot) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_i(\cdot)$, $\overline{\mathbb{E}}^0(\theta) = \theta$, and $\overline{\mathbb{E}}^t(\theta) = \overline{\mathbb{E}}\left[\overline{\mathbb{E}}^{t-1}(\theta)\right]$. This proof is in Lemma A.1. The second step is to show that Equation (DD.13) is a unique linear combination of the available signals in the economy. The second step of the proof is in Lemma A.2.

Lemma A.1. *There is a unique equilibrium in the first stage of the game, in which agent i's action a given by Equation (DD.13).*

Proof. We follow closely the proof of Proposition 1 in Hellwig and Veldkamp (2009), but adapting to our framework. The main difference is that in our setting there is a finite number of signals. Let $\hat{a}$ be the proposed equilibrium from Equation (DD.13) and let $\mathbb{A}$ be the set of functions $a_i = a(\mathbb{I}_i)$ such that

$$a(\mathbb{I}_i) = \int_{\omega}\left[(1-\tilde{r})b'\omega + \tilde{r}\frac{1}{n}\sum_{j=1}^{n}a(\mathbb{I}_j)\right]dF\left(\omega|\mathbb{I}_i\right),$$

where $\omega = (\theta, \varepsilon_1, \ldots, \varepsilon_n)'$ is a vector of i.i.d. standard normal random variables, $b = (1, 0, \ldots, 0)'$, and $F(\omega|\mathbb{I}_i)$ characterizes the distribution of ω given $\mathbb{I}_i$ as information set. We will show that $\tilde{a} \in \mathbb{A}$ if and only if $\tilde{a} = \hat{a}$ almost everywhere.

Let us define the functional $\mathcal{L}(\ldots)$ from L^2 to the real line:

$$\mathcal{L}(a) = \int_\omega \frac{1}{n}\sum_{i=1}^{n}\left[a(\mathbb{I}_i) - b'\omega\right]^2 dF(\omega) - \tilde{r}\int_\omega \left(\frac{1}{n}\sum_{i=1}^{n} a(\mathbb{I}_i) - b'\omega\right)^2 dF(\omega),$$

We proceed in two steps. First, we show that $\mathcal{L}(a)$ is strictly convex, and therefore if $\tilde{a}_1, \tilde{a}_2 \in \arg\min_a \mathcal{L}(a)$ then $\tilde{a}_1 = \tilde{a}_2$ almost everywhere. Second, we show that $\mathbb{A} = \arg\min_a \mathcal{L}(a)$, that is, $\tilde{a} \in \mathbb{A}$ if and only if $\tilde{a} \in \arg\min_a \mathcal{L}(a)$. Since $\hat{a} \in \mathbb{A}$, then $\hat{a}$ is unique except for measure zero perturbations.

First, we show that the functional $\mathcal{L}(a)$ is strictly convex. For any distinct functions $a_1(\mathbb{I}_i)$ and $a_2(\mathbb{I}_i)$, scalar $\alpha \in (0,1)$, and $\Delta(\mathbb{I}_i) \equiv a_2(\mathbb{I}_i) - a_1(\mathbb{I}_i)$, we have:

$$\begin{aligned}
&\mathcal{L}(\alpha a_1 + (1-\alpha)a_2) - \alpha\mathcal{L}(a_1) - (1-\alpha)\mathcal{L}(a_2) \\
&= \alpha\left[\mathcal{L}(a_1 + (1-\alpha)\Delta) - \mathcal{L}(a_1)\right] + (1-\alpha)\left[\mathcal{L}(a_2 - \alpha\Delta) - \mathcal{L}(a_2)\right] \\
&= \alpha\int_\omega \Bigg[(1-\alpha)^2\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)^2 + 2(1-\alpha)\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\left[a_1(\mathbb{I}_i) - b'\omega\right] \\
&\qquad -\tilde{r}(1-\alpha)^2\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2 - 2\tilde{r}(1-\alpha)\left(\frac{1}{n}\sum_{i=1}^{n}a_1(\mathbb{I}_i) - b'\omega\right)\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)\Bigg] dF(\omega) \\
&\quad + (1-\alpha)\int_\omega \Bigg[\alpha^2\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)^2 - 2\alpha\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\left[a_2(\mathbb{I}_i) - b'\omega\right] \\
&\qquad -\tilde{r}\alpha^2\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2 + 2\tilde{r}\alpha\left(\frac{1}{n}\sum_{i=1}^{n}a_2(\mathbb{I}_i) - b'\omega\right)\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)\Bigg] dF(\omega) \\
&= \int_\omega \Bigg[\Big(\alpha(1-\alpha)^2 + \alpha^2(1-\alpha) - 2\alpha(1-\alpha)\Big)\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)^2 \\
&\qquad - \tilde{r}\Big(\alpha(1-\alpha)^2 + \alpha^2(1-\alpha) - 2\alpha(1-\alpha)\Big)\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2\Bigg] dF(\omega) \\
&= -\alpha(1-\alpha)\int_\omega \left[\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)^2 - \tilde{r}\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2\right] dF(\omega) \\
&= -\alpha(1-\alpha)\int_\omega \left[\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2 + (1-\tilde{r})\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2\right] dF(\omega) \\
&= -\alpha(1-\alpha)\int_\omega \left[\frac{1}{n}\sum_{i=1}^{n}\left(\Delta(\mathbb{I}_i) - \frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2 + (1-\tilde{r})\left(\frac{1}{n}\sum_{i=1}^{n}\Delta(\mathbb{I}_i)\right)^2\right] dF(\omega) \le 0
\end{aligned}$$

The last inequality is strict if $\Delta(\mathbb{I}_i)$ is different from zero for a positive measure of events. Since $\mathcal{L}(a)$ is strictly convex, if $\tilde{a}_1, \tilde{a}_2 \in \arg\min_a \mathcal{L}(a)$ then $\tilde{a}_1 = \tilde{a}_2$ almost everywhere.

Next, we show that $\mathbb{A} = \arg\min_a \mathcal{L}(a)$. For any functions $a(\mathbb{I}_i)$ and $\delta(\mathbb{I}_i)$, and a scalar t, we have:

$$\mathcal{L}(a + t\delta) - \mathcal{L}(a) = t^2 A(\delta) + 2tB(a, \delta),$$

where

$$A(\delta) = \int_\omega \frac{1}{n}\sum_{i=1}^{n} \delta(\mathbb{I}_i)^2 - \tilde{r}\left(\frac{1}{n}\sum_{i=1}^{n}\delta(\mathbb{I}_i)\right)^2 dF(\omega)$$

$$B(a,\delta) = \int_\omega \frac{1}{n}\sum_{i=1}^{n} \delta(\mathbb{I}_i)\left[a(\mathbb{I}_i) - b'\omega\right] - \tilde{r}\left(\frac{1}{n}\sum_{i=1}^{n} a_1(\mathbb{I}_i) - b'\omega\right)\left(\frac{1}{n}\sum_{i=1}^{n}\delta(\mathbb{I}_i)\right) dF(\omega)$$

We have that $A(\delta) > 0$ whenever $\delta(\cdot)$ is different from zero for a positive measure. Therefore, $\mathcal{L}(a + t\delta)$ is minimized at $t^* = -\frac{B(a,\delta)}{A(\delta)}$ and $\mathcal{L}(a + t^*\delta) = \mathcal{L}(a) - \frac{B(a,\delta)^2}{A(\delta)}$. If $\tilde{a} \in \arg\min_a \mathcal{L}(a)$, then by the convexity of $\mathcal{L}(a)$ we have that $\tilde{a}$ is unique. Thus for any $\delta(\cdot)$, we have $B(\tilde{a},\delta) = 0$ since t^* minimizes $\mathcal{L}(a+t\delta)$ for any a and δ. If $\tilde{a} \in \arg\min_a \mathcal{L}(a)$ and $\tilde{a}' \notin \arg\min_a \mathcal{L}(a)$, then for $\delta = \tilde{a} - \tilde{a}'$ we have $B(\tilde{a}',\delta) \neq 0$. Therefore, $\tilde{a} \in \arg\min_a \mathcal{L}(a)$ if and only if $B(\tilde{a},\delta) = 0$ for every $\delta(\cdot)$.

We can write $B(a,\delta)$ as follows:

$$\begin{aligned} B(a,\delta) &= \int_\omega \frac{1}{n}\sum_{i=1}^{n} \delta(\mathbb{I}_i)\left[a(\mathbb{I}_i) - b'\omega\right] - \tilde{r}\left(\frac{1}{n}\sum_{i=1}^{n} a(\mathbb{I}_i) - b'\omega\right)\left(\frac{1}{n}\sum_{i=1}^{n}\delta(\mathbb{I}_i)\right) dF(\omega) \\ &= \frac{1}{n}\sum_{i=1}^{n}\int_\omega \delta(\mathbb{I}_i)\left[a(\mathbb{I}_i) - (1-\tilde{r})b'\omega - \tilde{r}\frac{1}{n}\sum_{j=1}^{n} a(\mathbb{I}_j)\right] dF(\omega) \end{aligned}$$

using law of iterated expectations,

$$\begin{aligned} &= \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}_i}\int_\omega \delta(\mathbb{I}_i)\left[a(\mathbb{I}_i) - (1-\tilde{r})b'\omega - \tilde{r}\frac{1}{n}\sum_{j=1}^{n} a(\mathbb{I}_j)\right] dF(\omega|\mathbb{I}_i)\, dF(\mathbb{I}_i) \\ &= \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}_i} \delta(\mathbb{I}_i)\left[a(\mathbb{I}_i) - \int_\omega (1-\tilde{r})b'\omega + \tilde{r}\frac{1}{n}\sum_{j=1}^{n} a(\mathbb{I}_j) dF(\omega|\mathbb{I}_i)\right] dF(\mathbb{I}_i) \end{aligned}$$

Notice that if $\tilde{a} \in \mathbb{A}$, then $B(\tilde{a},\delta) = 0$ for any $\delta(\cdot)$, using the definition of the set $\mathbb{A}$. This implies that $\tilde{a} \in \arg\min_a \mathcal{L}(a)$.

Finally, if $\tilde{a} \notin \mathbb{A}$, then by setting

$$\delta(\mathbb{I}_i) = \tilde{a}(\mathbb{I}_i) - \int_\omega (1-\tilde{r})b'\omega + \tilde{r}\frac{1}{n}\sum_{j=1}^{n}\tilde{a}(\mathbb{I}_j) dF(\omega|\mathbb{I}_i)$$

we have $\delta(\mathbb{I}_i) \neq 0$ and thus $B(a,\delta) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}_i}\delta(\mathbb{I}_i)^2 dF(\mathbb{I}_i) > 0$. As a result, $\tilde{a} \notin \arg\min_a \mathcal{L}(a)$. □

Lemma A.2. *Equation (DD.13) is a unique linear combination of the available signals in the economy.*

Proof. In this proof, we use the same notation as in the previous lemma: $\omega = (\theta, \varepsilon_1, \ldots, \varepsilon_n)'$ and $b = (1, 0, \ldots, 0)'$. Notice that we can write the signal structure of game in matrix notation as follows:

$$\underbrace{\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}_{n\times 1}}_{\equiv e} = \underbrace{\begin{bmatrix} 1 & \sigma_1 & 0 & \cdots & 0 \\ 1 & 0 & \sigma_2 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 1 & 0 & \cdots & 0 & \sigma_n \end{bmatrix}_{n\times n+1}}_{\equiv \Gamma} \underbrace{\begin{bmatrix} \theta \\ \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n+1\times 1}}_{\equiv \omega} \tag{DD.14}$$

or simply

$$e = \Gamma\omega$$

where the vector ω is a vector independent standard normal random variables.

However, player i only observes the signal e_j of players he is connected to, in addition to his own signal e_i. Thus, let the $\mathcal{K}_i + 1$ by n matrix X_i be the matrix that selects the signals observed by player i:

$$\mathbb{I}_i = \{e_j\}_{j=0:g_{ij}=1}^{n} = \begin{bmatrix} 0 \\ X_i e \end{bmatrix} = \begin{bmatrix} 0 \\ X_i \Gamma\omega \end{bmatrix} \tag{DD.15}$$

Using Bayes'updating rules,[1] we have that:

$$\mathbb{E}\left[\omega | \mathbb{I}_i\right] = \text{Cov}\left(\omega, X_i\Gamma\omega\right)' \, \text{Var}\left(X_i\Gamma\omega\right)^{-1} X_i\Gamma\omega$$

where

$$\begin{aligned} \text{Var}\left(X_i\Gamma\omega\right) &= X_i\Gamma\Gamma' X_i' \\ \text{Cov}\left(\omega, X_i\Gamma\omega\right) &= X_i\Gamma \end{aligned}$$

Let $\Delta_i = \Gamma' X_i' \left(X_i\Gamma\Gamma' X_i'\right)^{-1} X_i\Gamma$, we can write

$$\mathbb{E}\left[\omega | \mathbb{I}_i\right] = \Delta_i\omega$$

Thus, we have

$$\overline{\mathbb{E}}(\omega) = \frac{1}{n}\sum_{i=1}^{n} \Delta_i\omega = \overline{\Delta}\omega,$$

where $\overline{\Delta} = \frac{1}{n}\sum_{i=1}^{n}\Delta_i$. Notice that Δ_i is idempotent and thus its eigenvalues are either zero or one. Since $\frac{1}{n}\Delta_i$ is symmetric with eigenvalues between zero and $\frac{1}{n}$, then the eigenvalues of $\overline{\Delta}$ are between zero and one.[2] Furthermore, we have that $\overline{\mathbb{E}}^0(\omega) = \omega$, $\overline{\mathbb{E}}^1(\omega) = \overline{\Delta}\omega$, $\overline{\mathbb{E}}^2(\omega) = \overline{\Delta}^2\omega$, and, more generally,

$$\overline{\mathbb{E}}^t(\omega) = \overline{\Delta}^t\omega.$$

Hence, using $b = (1, 0, \ldots, 0)'$, we can write Equation (DD.13) as follows:

$$\begin{aligned} a(\mathbb{I}_i) &= (1-\tilde{r})\sum_{t=1}^{\infty} \tilde{r}^t \mathbb{E}_i\left[\overline{\mathbb{E}}^t(\theta)\right] \\ &= (1-\tilde{r})\sum_{t=1}^{\infty} \tilde{r}^t \mathbb{E}_i\left[\overline{\mathbb{E}}^t(b'\omega)\right] \\ &= (1-\tilde{r})b'\sum_{t=1}^{\infty} \tilde{r}^t \mathbb{E}_i\left[\overline{\Delta}^t\omega\right] \end{aligned}$$

[1] See Hellwig and Veldkamp (2009).

[2] See Theorem 1 in Thompson and Freede (1971).

$$= (1-\tilde{r})b' \sum_{t=1}^{\infty} \tilde{r}^t \overline{\Delta}^t \mathbb{E}_i[\omega]$$
$$= (1-\tilde{r})b' \sum_{t=1}^{\infty} \tilde{r}^t \overline{\Delta}^t \Delta_i \omega.$$

Since the largest eigenvalue of $\overline{\Delta}$ in absolute value is less than or equal to one, the eigenvalues of $\tilde{r}\overline{\Delta}$ are stricly less than one in absolute value. Thus the limit is unique and given by:

$$a(\mathbb{I}_i) = (1-\tilde{r})\tilde{r}b' \left[I - \tilde{r}\overline{\Delta}\right]^{-1} \overline{\Delta}\Delta_i \omega.$$

□

A.2 Proof of Proposition 2

In this subsection, we derive the payoff function under the optimal action.

The payoff function of player i net of link formation costs is given by

$$U_i = -\left(a_i - a_i^*\right)^2$$

where $a_i^* = (1-r_i)\theta + r_i\overline{a}_{-i}$ is player i's bliss action and $\overline{a}_{-i} = \frac{1}{n-1}\sum_{j=1\, j\neq i}^{n} a_j = \sum_{j=1}^{n} \beta_{-i,j} e_j$ is the average action without i's own action.[3] Notice that player i takes all $\beta_{-i,j}$'s as given.

Using the notation from Equations (DD.14) and (DD.15), notice that the optimal action of player i is $a_i = \mathbb{E}\left[a_i^* | \mathbb{I}_i\right]$, and the expected payoff net of link formation costs for a given network G conditional on the common prior is given by

$$\mathbb{E}\left[U_i | G\right] = -\mathbb{E}\left[\left(a_i - a_i^*\right)^2 | G\right] = -\mathbb{E}\left[\mathbb{E}\left[\left(a_i - a_i^*\right)^2 | \mathbb{I}_i\right] | G\right]$$

where the last equality hold based on law of iterated expectations. Using the optimal action choice, the expected value conditional on player i's informational set is a conditional variance. Hence, the payoff function net of link formation costs is further simplified to

$$\mathbb{E}\left[U_i | G\right] = -\mathbb{E}\left[\mathrm{Var}\left(a_i^* | \mathbb{I}_i\right) | G\right]$$

Next, let's write the bliss action in matrix notation:

$$\begin{aligned}
a_i^* &= (1-r_i)\theta + r_i\overline{a}_{-i} \\
&= (1-r_i)\theta + r_i \sum_{j=1}^{n} \beta_{-i,j} e_j \\
&= (1-r_i)\theta + r_i \sum_{j=1}^{n} \beta_{-i,j}\theta + r_i \sum_{j=1}^{n} \beta_{-i,j}\sigma_j \varepsilon \\
&= (1-r_i\beta_{-i,0})\theta + r_i \sum_{j=1}^{n} \beta_{-i,j}\sigma_j \varepsilon_j \\
a_i^* &= \underbrace{\left[1 - r_i\beta_{-i,0},\ r_i\sigma_1\beta_{-i,1},\ r_i\sigma_2\beta_{-i,2},\ \cdots,\ r_i\sigma_n\beta_{-i,n}\right]}_{\equiv F_i'} \omega = F_i'\omega
\end{aligned} \tag{DD.16}$$

[3] Remember that $e_0 = 0$, so we could have defined $\overline{a}_{-i} = \sum_{j=0}^{n} \beta_{-i,j} e_j$ instead.

where $\beta_{-i,0} = 1 - \sum_{j=1}^{n} \beta_{-i,j}$. The vector F_i does not depend on player i observed signal, it only depends on the network itself. Additionally, we can use Bayes updating rule to represent the optimal action as follows:

$$a_i = \mathbb{E}\left[a_i^*\right] = \mathbb{E}\left[F_i' \omega | \mathbb{I}_i\right] = F_i' \mathbb{E}\left[\omega | \mathbb{I}_i\right], \tag{DD.17}$$

where $\mathbb{E}\left[\omega | \mathbb{I}_i\right] = \text{Cov}\left(\omega, X_i \Gamma \omega\right)' \text{Var}\left(X_i \Gamma \omega\right)^{-1} X_i \Gamma \omega$.

Player i's expected payoff net of link formation costs becomes

$$\mathbb{E}\left[U_i | G\right] = -\mathbb{E}\left[\text{Var}\left(a_i^* | \mathbb{I}_i\right) | G\right] = -F_i' \text{Var}\left(\omega | \mathbb{I}_i\right) F_i \tag{DD.18}$$

and we can use Bayes' updating rule to compute the variance covariance term:

$$\text{Var}\left(\omega | \mathbb{I}_i\right) = \text{Var}\left(\omega\right) - \text{Cov}\left(\omega, X_i \Gamma \omega\right)' \text{Var}\left(X_i \Gamma \omega\right)^{-1} \text{Cov}\left(\omega, X_i \Gamma \omega\right) \tag{DD.19}$$

where

$$\begin{aligned} \text{Var}\left(\omega\right) &= \text{I} \\ \text{Var}\left(X_i \Gamma \omega\right) &= X_i \Gamma \Gamma' X_i' \\ \text{Cov}\left(\omega, X_i \Gamma \omega\right) &= X_i \Gamma \end{aligned}$$

In order to successfully invert the variance-covariance matrix $\text{Var}\left(X_i \Gamma \omega\right)$, let's redefine Γ as

$$\Gamma = \begin{bmatrix} \mathbf{1} & \Phi \end{bmatrix}$$

where $\mathbf{1}$ is a column vector of ones and

$$\Phi = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix}_{n \times n} \tag{DD.20}$$

Using the above notation, we can simplify $\text{Var}\left(X_i \Gamma \omega\right)$ as follows

$$\text{Var}\left(X_i \Gamma \omega\right) = X_i \Gamma \Gamma' X_i' = X_i \Phi \Phi' X_i' + \mathbf{1}\mathbf{1}'$$

Notice that $X_i \Phi \Phi' X_i'$ is a diagonal matrix variance σ_j^2's of signals that player i observes. This simplification is useful because $X_i \Phi \Phi' X_i'$ is easy to invert and we can apply Shermann-Morrison theorem:[4]

$$\text{Var}\left(X_i \Gamma \omega\right)^{-1} = \left(X_i \Phi \Phi' X_i'\right)^{-1} - \frac{1}{\phi_i} \left(X_i \Phi \Phi' X_i'\right)^{-1} \mathbf{1}\mathbf{1}' \left(X_i \Phi \Phi' X_i'\right)^{-1} \tag{DD.21}$$

where $\phi_i = 1 + \mathbf{1}' (X_i \Phi \Phi' X_i')^{-1} \mathbf{1} = 1 + \sum_{j=1}^{n} g_{ij} \sigma_j^{-2}$. We can use the simplified inverse of the variance to compute the

[4] For any non-singular matrix A, column vectors u and v, and a scalar α, Shermann-Morrison theorem states that

$$\left(A + \beta u v'\right)^{-1} = A^{-1} - \frac{\alpha}{\phi} A^{-1} u v' A^{-1}$$

where $\phi = 1 + \alpha v' A^{-1} u$. We apply this result by setting $A = X_i \Phi \Phi' X_i'$, $\alpha = 1$, $u = v = \mathbf{1}$. See Golub and Van Loan (2012) for more details.

following:

$$\text{Cov}(\omega, X_i\Gamma\omega)' \,\text{Var}(X_i\Gamma\omega)^{-1}\,\text{Cov}(\omega, X_i\Gamma\omega) = \begin{bmatrix} \mathbf{1}' \\ \Phi' X_i' \end{bmatrix} \text{Var}(X_i\Gamma\omega)^{-1} \begin{bmatrix} \mathbf{1} & X_i\Phi \end{bmatrix}$$

$$= \mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$$

where

$$\mathcal{A}_{11} = \mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1} - \tfrac{1}{\phi_i}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}$$

$$\mathcal{A}_{12} = \mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi - \tfrac{1}{\phi_i}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi$$

$$\mathcal{A}_{21} = \Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1} - \tfrac{1}{\phi_i}\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}$$

$$\mathcal{A}_{22} = \Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi - \tfrac{1}{\phi_i}\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi$$

We can use $\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1} = \phi_i - 1$ to get

$$\mathcal{A} = \begin{bmatrix} \frac{\phi_i - 1}{\phi_i} & \frac{1}{\phi_i}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi \\ \frac{1}{\phi_i}\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1} & \mathcal{A}_{22} \end{bmatrix}$$

Notice that

$$\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1} = \begin{bmatrix} \frac{g_{i1}}{\sigma_1} \\ \frac{g_{i2}}{\sigma_2} \\ \vdots \\ \frac{g_{in}}{\sigma_n} \end{bmatrix}_{n\times 1}$$

$$\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi = \begin{bmatrix} g_{i1} & 0 & \cdots & 0 \\ 0 & g_{i2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & g_{in} \end{bmatrix}_{n\times n}$$

$$\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i\Phi = \begin{bmatrix} \frac{g_{i1}g_{i1}}{\sigma_1\sigma_1} & \frac{g_{i1}g_{i2}}{\sigma_1\sigma_2} & \cdots & \frac{g_{i1}g_{in}}{\sigma_1\sigma_n} \\ & \vdots & & \\ \frac{g_{in}g_{i1}}{\sigma_n\sigma_1} & \frac{g_{in}g_{i2}}{\sigma_n\sigma_2} & \cdots & \frac{g_{in}g_{in}}{\sigma_n\sigma_n} \end{bmatrix}_{n\times n}$$

As a result, the matrix $\mathcal{A}$ is further simplified to

$$\mathcal{A} = \frac{1}{\phi_i}\begin{bmatrix} \phi_i - 1, & \frac{g_{i1}}{\sigma_1}, & \frac{g_{i2}}{\sigma_2}, & \cdots & \frac{g_{in}}{\sigma_n} \\ \frac{g_{i1}}{\sigma_1}, & g_{i1}\phi_i - \frac{g_{i1}g_{i1}}{\sigma_1\sigma_1}, & -\frac{g_{i1}g_{i2}}{\sigma_1\sigma_2}, & \cdots & -\frac{g_{i1}g_{in}}{\sigma_1\sigma_n} \\ \frac{g_{i2}}{\sigma_2}, & -\frac{g_{i2}g_{i1}}{\sigma_2\sigma_1}, & \phi_i g_{i2} - \frac{g_{i2}g_{i2}}{\sigma_2\sigma_2}, & \cdots & -\frac{g_{i2}g_{in}}{\sigma_2\sigma_n} \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{g_{in}}{\sigma_n}, & -\frac{g_{in}g_{i1}}{\sigma_n\sigma_1}, & -\frac{g_{in}g_{i2}}{\sigma_n\sigma_2}, & \cdots & g_{in} - \frac{g_{in}g_{in}}{\sigma_n\sigma_n} \end{bmatrix}$$

and player i's expected payoff net of link formation costs becomes

$$\begin{aligned}
\mathbb{E}\left[U_i|G\right] &= -F_i'\left(\mathrm{I} - \mathcal{A}\right)F_i \\
&= -\frac{1}{\phi_i}F_i'\begin{bmatrix} 1, & -\frac{g_{i1}}{\sigma_1}, & -\frac{g_{i2}}{\sigma_2}, & \cdots & -\frac{g_{in}}{\sigma_n} \\ -\frac{g_{i1}}{\sigma_1}, & (1-g_{i1})\phi_i + \frac{g_{i1}}{\sigma_1\sigma_1}, & \frac{g_{i1}g_{i2}}{\sigma_1\sigma_2}, & \cdots & \frac{g_{i1}g_{in}}{\sigma_1\sigma_n} \\ -\frac{g_{i2}}{\sigma_2}, & \frac{g_{i2}g_{i1}}{\sigma_2\sigma_1}, & (1-g_{i2})\phi_i + \frac{g_{i2}}{\sigma_2\sigma_2}, & \cdots & \frac{g_{i2}g_{in}}{\sigma_2\sigma_n} \\ \vdots & \vdots & & \ddots & \vdots \\ -\frac{g_{in}}{\sigma_n}, & \frac{g_{in}g_{i1}}{\sigma_n\sigma_1}, & \frac{g_{in}g_{i2}}{\sigma_n\sigma_2}, & \cdots & (1-g_{in})\phi_i + \frac{g_{in}}{\sigma_n\sigma_n} \end{bmatrix}F_i \\
&= -\frac{1}{\phi_i}\begin{bmatrix} 1 - r_i\sum_{j=0} g_{ij}\beta_{-i,j} \\ -g_{i1}\frac{1}{\sigma_1} + g_{i1}r_i\frac{1}{\sigma_1}\sum_{j=0}^n g_{ij}\beta_{-i,j} + (1-g_{i1})\phi_i r_i\beta_{-i,1}\sigma_1 \\ \vdots \\ -g_{in}\frac{1}{\sigma_n} + g_{in}r_i\frac{1}{\sigma_n}\sum_{j=0}^n g_{ij}\beta_{-i,j} + (1-g_{in})\phi_i r_i\beta_{-i,n}\sigma_n \end{bmatrix}' F_i \\
&= -\frac{1}{\phi_i}\Bigg\{(1 - r_i\beta_{-i,0})(1 - r_i\sum_{j=0} g_{ij}\beta_{-i,j}) \\
&\qquad -r_i\sum_{s=1}^n \beta_{-i,s}g_{is}(1 - r_i\sum_{j=0} g_{ij}\beta_{-i,j}) + \phi_i r_i^2\sum_{j=1}^n(1-g_{ij})\beta_{-i,j}^2\sigma_j^2\Bigg\} \\
\mathbb{E}\left[U_i|G\right] &= -\frac{1}{\phi_i}(1 - r_i\sum_{j=0} g_{ij}\beta_{-i,j})^2 - r_i^2\sum_{j=0}^n(1-g_{ij})\beta_{-i,j}^2\sigma_j^2 \qquad \text{(DD.22)}
\end{aligned}$$

where $\phi_i = 1 + \sum_{j=1}^n g_{ij}\sigma_j^{-2}$.

If $r_i = r$ for every $i = 1,\ldots,n$ and $\sigma_j = \sigma$ for every $j = 1,\ldots,n$, then $\phi_i = \frac{\sigma^2+\mathcal{K}_i+1}{\sigma^2}$ and the expected payoff including link formation costs becomes

$$-\frac{\sigma^2}{\sigma^2+\mathcal{K}_i+1}\left(1 - r\sum_{j=0} g_{ij}\beta_{-i,j}\right)^2 - r^2\sigma^2\sum_{j=0}^n(1-g_{ij})\beta_{-i,j}^2 - c\left(\mathcal{K}_i\right)$$

where $\mathcal{K}_i = \sum_{j=1,j\neq i}^{n} g_{ij}$. The is exactly the payoff expression in Proposition 2.

Furthermore, if $\sigma = 1$ then the expected payoff is given by

$$-\frac{1}{\mathcal{K}_i + 2}\left(1 - r\sum_{j=0} g_{ij}\beta_{-i,j}\right)^2 - r^2\sum_{j=0}^{n}(1 - g_{ij})\beta_{-i,j}^2 - c(\mathcal{K}_i)$$

B Equilibrium Properties

In Section B.1, we prove Proposition 3 by showing that any equilibrium satisfies properties 1 and 2. In Section B.2, we show an example of an out-of-equilibrium violation of these properties. In Section B.3, we prove Proposition 4, and in Section B.4 an example of an equilibrium with a three-tier information structure.

B.1 Proof of Proposition 3

B.1.1 Property 1

We start by showing that given Assumption 1, any linear action-strategy strict Nash equilibrium of the game above satisfies Property 1.

The argument works in two main steps. First, we show that an agent's best response to other agents' choices of connections is to observe the signal of the most influential agent. In Lemma A.3, we show that agent i's best response to other agents' choices of connections is to observe the signal of the most influential agent, where agents are ranked by a centrality measure that is specific to agent i, $\beta_{-i,\cdot}$. While in Lemma A.4 we show that, in equilibrium, all agents rank which signal to observe in the same way. That is, we show that the agent specific ranking $\beta_{-i,\cdot}$ coincides in equilibrium for all agents, and is captured by the vector of influences over the average action β. Second, we show that an agent whose signal is more observed is also the one that has a more influential signal. In Lemma A.6 we show that more people observe agent m's signal than agent l's signal if, and only if, agent m's signal is more influential for the average action than agent l's signal. This guarantees that, in equilibrium, the ranking implied by influence is the same ranking implied by the number of agents observing a signal.

Let us start by showing the monotonicity of best responses. We show that for any agent i, set of connections of i g_i, set of connections of other agents, and other agents linear action strategies, agent i's best response dictates that if she finds optimal to observe another player's signal, then she observes any other more influential signal as well.

Lemma A.3. *For any strategy played by the other agents, in any best response by agent i, if $i \neq l$ and $g_{i,l} = 1$, then $g_{i,m} = 1$ for any signal m such that $\beta_{-i,m} > \beta_{-i,l}$.*

Furthermore, if the best response is strict, then if $i \neq l$ and $g_{i,l} = 1$, then $g_{i,m} = 1$ for any signal m such that $\beta_{-i,m} \geq \beta_{-i,l}$.

Proof. Observe agent i's expected payoff formula from Proposition 2. First, note that player i's connection decisions or linear action decisions do not influence the vector of influences given by $\beta_{-i,\cdot}$. Thus, an agent connections only affect her payoff through the g_i. Finally, observe that player's i payoff derivative with respect to $\beta_{-i,l}$ is strictly positive for any $g_{i,l} = 1$, and strictly negative for $g_{i,l} = 0$. This completes the proof. □

We now proceed to the second part of our argument. We establish that the agent, j, whose signal is more observed is also the agent j with higher $\beta_{-i,j}$, for all i.

The first step is to establish a relationship between $\beta_{-i,j}$ and β_j in any strict equilibrium.

Lemma A.4. *For any agent i, given a strategy played by the other agents, in any strict best response by i, $\beta_m \geq \beta_l$ implies that $\beta_{-i,m} \geq \beta_{-i,l}$. Furthermore, if $\beta_m > \beta_l$, then we have $\beta_{-i,m} > \beta_{-i,l}$*

Proof. We proceed by exhaustion. Agent i's connections must satisfy one of the following situations: (i) $g_{im} = 1$ and $g_{il} = 0$; (ii) $g_{im} = g_{il} = 0$; (iii) $g_{im} = g_{il} = 1$; or (iv) $g_{im} = 0$ and $g_{il} = 1$. We start with the first case. If $g_{if} = 1$ and $g_{ih} = 0$, it must be that $\beta_{-i,f} > \beta_{-i,h}$ by Lemma A.3.

For the other 3 cases, for a general pair i and j and using Equation (DD.11), the loading $\beta_{-i,j}$ can be written as

$$\beta_{-i,j} = \frac{n}{n-1}\beta_j - \frac{1}{n-1}\lambda_{ij},$$

where, as shown in Equation (DD.6),

$$\begin{aligned}
\lambda_{ij} &= (1-\tilde{r})\frac{g_{ij}}{\mathcal{K}_i + 1{+}\sigma^2} + \tilde{r}g_{ij}\beta_j + \tilde{r}\frac{g_{ij}}{\mathcal{K}_i + 1{+}\sigma^2} - \tilde{r}\sum_{s=0}^{n}\beta_s g_{is}\frac{g_{ij}}{\mathcal{K}_i + 1{+}\sigma^2} \\
&= \frac{g_{ij}}{\mathcal{K}_i + 1{+}\sigma^2} + \tilde{r}\left[g_{ij}\beta_j - \sum_{s=0}^{n}\beta_s g_{is}\frac{g_{ij}}{\mathcal{K}_i + 1{+}\sigma^2}\right] \\
&= g_{ij}\left[\frac{1}{\mathcal{K}_i + 1{+}\sigma^2} + \tilde{r}\left[\beta_j - \frac{\sum_{s=0}^{n}\beta_s g_{is}}{\mathcal{K}_i + 1{+}\sigma^2}\right]\right]
\end{aligned}$$

is the weight agent i puts to the j^{th} signal on her optimal linear action.

Substituting in the $\beta_{-i,j}$ expression,

$$\begin{aligned}
\beta_{-i,j} &= \frac{n}{n-1}\beta_j - \frac{1}{n-1}\lambda_{ij} \\
&= \frac{n}{n-1}\beta_j - \frac{g_{ij}}{n-1}\left[\frac{1}{\mathcal{K}_i + 1{+}\sigma^2} + \tilde{r}\left[\beta_j - \frac{\sum_{s=0}^{n}\beta_s g_{is}}{\mathcal{K}_i + 1{+}\sigma^2}\right]\right] \\
&= \frac{n}{n-1}\beta_j - \frac{g_{ij}}{n-1}\left[\frac{1}{\mathcal{K}_i + 1{+}\sigma^2} + \tilde{r}\left[\beta_j - \frac{\sum_{s=0}^{n}\beta_s g_{is}}{\mathcal{K}_i + 1{+}\sigma^2}\right]\right]
\end{aligned}$$

Applying the above to our three remaining cases gives us:

If $g_{im} = g_{il} = 0$, then $\beta_{-i,m} - \beta_{-i,l} = \frac{n}{n-1}(\beta_m - \beta_l) \geq 0$.

If $g_{im} = g_{il} = 1$, then $\beta_{-i,m} - \beta_{-i,l} = \left(\frac{n}{n-1} - \frac{\tilde{r}}{n-1}\right)(\beta_m - \beta_l) \geq 0$.

If $g_{im} = 0$ and $g_{il} = 1$, then $\beta_{-i,m} - \beta_{-i,l} = \frac{n}{n-1}(\beta_m - \beta_l) + \frac{1-\tilde{r}\sum_{s=0}^{n}\beta_s g_{is}}{(n-1)(\mathcal{K}_i+1+\sigma^2)} + \frac{\tilde{r}}{n-1}\beta_l > 0$.

Thus, we have that $\beta_m \geq \beta_l \implies \beta_{-i,m} \geq \beta_{-i,l}$ and $\beta_m > \beta_l \implies \beta_{-i,m} > \beta_{-i,l}$. □

To show that in any strict Nash equilibrium Property 1 is satisfied, all what's left to show is that an agent whose signal is more observed has a higher influence on the average action in any strict equilibrium, that is $\overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l \implies \beta_m \geq \beta_l$. Before that, we show a simple property about cross-looks: If a higher ranked agent observes the signal of a lower ranked agent, then in equilibrium the lower ranked agent also observes the higher ranked agent signal. Although in content the following Lemma is closer to Property 2, its proof is a lot simpler. To prove the lemma, we present a revealed preference argument. It suffices in this case (and not for Property 2) as we only have to compare one player's deviation at a time. We use this result in the proof of Lemma A.6.

Lemma A.5. *Let agents h and f be ranked by their influence over the average action, such that $\beta_f < \beta_h$. If in equilibrium $g_{h,f} = 1$, then $g_{f,h} = 1$.*

Proof. The proof proceeds by contradiction. Assume that in a proposed equilibrium, $g_{h,f} = 1$ and $g_{f,h} = 0$.

Let Π_f and Π_h be the players' equilibrium payoffs. Since $g_{h,f} = 1$, agent f can simply copy agent h's connections if she wanted to—and also copy the linear action weights—which implies that $\Pi_f \geq \Pi_h$ by revealed preference. It is worth noting that if agent f copied agent h's connections they would both have the same information set. This guarantees that they could have the same payoff, as well as same linear action coefficients.

Let $\hat{\Pi}_f$ be the payoff of agent f if agent f stops observing her own signal and observes player h signal for free. By the monotonicity of agent i's expected payoff formula, given in Proposition 2, and Lemma A.4, $\beta_{-f,h} > \beta_{-f,f}$, and thus $\hat{\Pi}_f > \Pi_f$.

Finally, if agent h observes the same set of signals as agent f does in the proposed equilibrium, (except that she observes her own signal and does not observe agent f's signal), her payoff cannot be less than $\hat{\Pi}_f$. Agent h could simply copy the connections and linear action weights used by agent f to obtain the payoff $\hat{\Pi}_f$. By revealed preference, this gives us $\hat{\Pi}_f \leq \Pi_h$. A contradiction. □

Lemma A.6. *Agent m's signal has a weakly higher impact in the average action than agent l's has if, and only if, agent m is weakly more observed than agent l.*

$$\beta_m \geq \beta_l \iff \overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l$$

Proof. First of all, note that lemmata A.3 and A.4 guarantees one side of the argument, that if $\beta_m \geq \beta_l$ then $\overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l$. However, we need to show the other direction to guarantee Property 1. Thus, assume not. That is, assume that $\beta_m < \beta_l$ and at the same time $\overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l$.

From the fact that all players use the beta ranking to decide who to look, we have that $\beta_l > \beta_m$ implies $\beta_{-i,l} > \beta_{-i,m}$ $\forall i$, and thus $g_{i,l} \geq g_{i,m}$ $\forall i \neq f$.

Let us now proceed in cases. There are two possible cases, (i) $g_{l,m} = 1$, which gives us $g_{m,l} = 1$ by Lemma A.5, or (ii) $g_{l,m} = 0$, which gives us $g_{m,l} = 0$ otherwise $\overline{\mathcal{K}}_l > \overline{\mathcal{K}}_m$. So, in both cases, we have that $g_{l,m} = g_{m,l}$.

Since $g_{i,l} \geq g_{i,m}$ $\forall i \neq m$ and $\overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l$, it must be that $\overline{\mathcal{K}}_m = \overline{\mathcal{K}}_l$, which implies that $g_{i,l} = g_{i,m}$ $\forall i$.

Let's now consider the formula for the influence of a signal j, β_j from Equation (DD.9).

$$\beta_j = \frac{1}{n}\sum_{i=1}^{n}\frac{g_{ij}}{\sigma^2 + \mathcal{K}_i + 1} + \frac{\tilde{r}}{n}\left[\beta_j\left(\overline{\mathcal{K}}_j + 1\right) - \sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is} g_{ij}}{\sigma^2 + \mathcal{K}_i + 1}\right].$$

Applying this formula to both signals m and l, and computing the difference:

$$\begin{aligned}\beta_m - \beta_l &= \frac{1}{n}\sum_{i=1}^{n}\frac{g_{im} - g_{il}}{\sigma^2 + \mathcal{K}_i + 1} + \frac{\tilde{r}}{n}\left[\beta_m(\overline{\mathcal{K}}_m + 1) - \beta_l(\overline{\mathcal{K}}_l + 1) - \sum_{i=1}^{n}\sum_{s=0}^{n}\frac{\beta_s g_{is}(g_{im} - g_{il})}{\sigma^2 + \mathcal{K}_i + 1}\right] \\ &= \frac{\tilde{r}}{n}\left[(\beta_m - \beta_l)(\overline{\mathcal{K}}_m + 1)\right]\end{aligned}$$

Given that $\tilde{r}\left(\overline{\mathcal{K}}_m + 1\right) < n$, we must have $\beta_m - \beta_l = 0$, a contradiction. □

Finally, to show that any linear action-strategy strict Nash equilibrium of the game satisfies Property 1 all what is left is to use the Lemmas above. In any strict equilibrium, by Lemma A.6, if $\overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l$ we have that $\beta_m \geq \beta_l$. Lemma A.4, shows that $\beta_m \geq \beta_l$ guarantees $\beta_{-i,m} \geq \beta_{-i,l}$, and finally by Lemma A.3 $\beta_{-i,m} \geq \beta_{-i,l}$ implies that for any $l \neq i$, $g_{i,l} = 1 \implies g_{i,m} = 1$.

B.1.2 Property 2

The proof that Property 2 holds in equilibrium is a little more evolved. Before discussing the details of the proof, let us define two sets, D_M and D_L, for two players m and l with

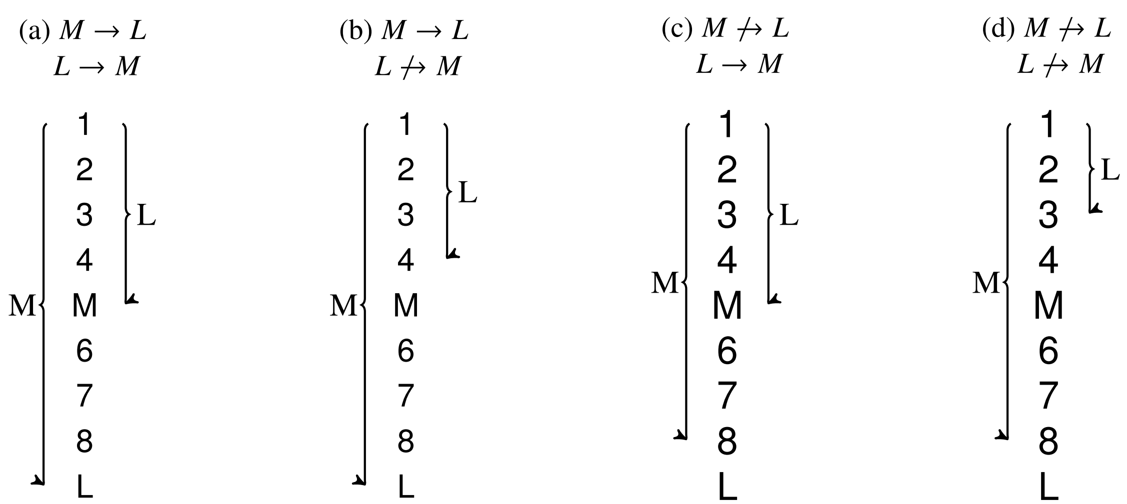
$$\mathcal{K}_m > \mathcal{K}_l.$$

By lemmata A.3 and A.4, all players are ranked according to a common list and thus all signals that player l pays to observe, player m also observes.

Let there be $d = \mathcal{K}_m - \mathcal{K}_l \geq 1$ signals. Abusing notation, we call the corresponding set of signals, D_M and D_L defined as follows: D_M is the set of d signals that agent m is currently observing but would stop observing if agent m were to observe d fewer signals. Similarly, D_L is the set of d signals that agent l is not currently observing but would start observing if agent l were to observe d additional signals. Accordingly, if l were to form connections to D_L she would be obtaining the same number of signals as m. Note that we define D_M and D_L to be the set of signals that give the best possible information set for players m and l that satisfy the above.

The fact that player m receives signal m for free disciplines the sets D_M and D_L. They are not equal, as agent m cannot deviate and stop observing her own signal. Let us also define the set S_M, to be the set of signals that agent m observes and l does not observe, while the set S_L is the set of signals that agent l observes and m does not.

Example 1. *Consider the following example, in which player m is the* 5^{th} *most attractive signal to be tapped into while player l is the* 9^{th}*. In each of the four situations above, we contemplate a different configuration between the two players. In the first, they both tap into each other signals, while on the last neither does so. In the second, even though m taps into l′s signal, l does not correspond, and finally, the third situation presents the inverse.*



$D_M = \{7, 8, L\}$, $D_L = \{6, 7, 8\}$, $d = 3$

$D_M = \{6, 7, 8, L\}$, $D_L = \{M, 6, 7, 8\}$, $d = 4$

$D_M = \{7, 8\}$, $D_L = \{6, 7\}$, $d = 2$

$D_M = \{4, 6, 7, 8\}$, $D_L = \{4, M, 6, 7\}$, $d = 4$

Figure 1: Four different configurations concerning players m and l choices of connections.

The four examples pictured above show different possibilities for the sets D_M *and* D_L*, depending on the connections formed. It is also interesting to understand what is the information set of players in each situation. In the first* *situation, the players Information sets are* $I_M = \{e_0, e_1, e_2, e_3, e_4, e_M, e_6, e_7, e_8, e_L\}$ *and* $I_L = \{e_0, e_1, e_2, e_3, e_4, e_M, e_L\}$, *and thus* $S_M = \{e_6, e_7, e_8\}$ *and* $S_L = \{\emptyset\}$. *In the second situation, they are* $I_M = \{e_0, e_1, e_2, e_3, e_4, e_M, e_6, e_7, e_8, e_L\}$ *and* $I_L = \{e_0, e_1, e_2, e_3, e_4, e_L\}$ *(and thus* $S_M = \{e_M, e_6, e_7, e_8\}$ *and* $S_L = \{\emptyset\}$*), while on the third they are* $I_M = \{e_0, e_1, e_2, e_3, e_4, e_M, e_6, e_7, e_8\}$ *and* $I_L = \{e_0, e_1, e_2, e_3, e_4, e_M, e_L\}$, *with* $S_M = \{e_6, e_7, e_8\}$ *and* $S_L = \{e_L\}$. *Finally, in the fourth configuration,* $I_M = \{e_0, e_1, e_2, e_3, e_4, e_M, e_6, e_7, e_8\}$ *and* $I_L = \{e_0, e_1, e_2, e_3, e_L\}$, *and thus* $S_M = \{e_4, e_M e_6, e_7, e_8\}$ *and* $S_L = \{e_L\}$.

The example above highlights comparative properties of the sets D_M and D_L. The signals listed in D_L weakly dominate the ones listed in D_M. This is a direct result of the fact that player m receives the signal m for free (and m is more attractive than l) and cannot stop observing it, while player l receives signal l.

We will prove Property 2 by contradiction. The general structure of the argument is to assume that in equilibrium an agent m is at the same time strictly observing more signals and has her signal observed more than another player l, that is:

$$\overline{\mathcal{K}}_m \geq \overline{\mathcal{K}}_l \text{ and } \mathcal{K}_m > \mathcal{K}_l$$

The contradiction will be constructed in the following way: If it is worth for agent m to pay and observe more signals than l does (even though agent $m's$ free signal is more observed than agent l's free signal), then agent l's deviation to look at those signals is profitable.

The argument of the proof is to show that if player m is not willing to deviate and stop observing signals in D_M, then it is optimal for agent l to deviate and start looking at players in D_L. However, we do not proceed directly to it. We first show that if player m is not willing to deviate and stop looking at signals D_L, then it would be optimal for agent l to deviate and start looking at players in D_L. However, by the fact that the set D_L weakly dominates the set D_M, and by Lemma A.3 and Lemma A.4, we know that the deviation to stop observing signals in D_L is weakly dominated by the original deviation to stop observing the signals in D_M. From now on, we call the set D_L as D, for deviation.

Before we proceed with the analysis, let us show the two lemmas below hold for the set D.

Lemma A.7. *For a given non-empty set of signals D and two distinct agents l and m such that m observe all signals in D while agent l does not observe signals in D, the summed influence of the signals in the set D is higher if the average action excludes agent l's action than if it excludes agent m's action:*

$$\beta_{-l,j} > \beta_{-m,j} \textit{ for every } j \in D.$$

Proof. This is a direct result of the fact that l is not observing any signal in D while m is observing signals of D. This implies that l's action cannot respond to signals in D, i.e., $\lambda_{lj} = 0 \, \forall \, j \in D$. Using the expressions for $\beta_{-m,j}$ and $\beta_{-l,j}$ from Equation (DD.11), we have that:

$$\beta_j = \frac{n-1}{n}\beta_{-m,j} + \frac{1}{n}\lambda_{m,j} = \frac{n-1}{n}\beta_{-l,j} + \frac{1}{n}\lambda_{l,j} = \frac{n-1}{n}\beta_{-l,j}$$

Hence,

$$\beta_{-m,j} = \beta_{-l,j} - \frac{\lambda_{m,j}}{n-1} < \beta_{-l,j},$$

since $\lambda_{mj} > 0$ whenever $g_{mj} = 1$, i.e., for every $j \in D$, as discussed in Appendix A.1.2 (Equation DD.5). □

Lemma A.8. *For a given non-empty set of signals D and two distinct agents l and m such that m observe all signals* *in D while agent l does not observe signals in D, if* $\sum_{j=0}^{n}(g_{mj} - g_{lj})\beta_j \geq \sum_{j \in D} \beta_j$*, then:*

$$\sum_{j \notin D} g_{m,j}\beta_{-m,j} - \sum_{j \notin D} g_{l,j}\beta_{-l,j} \geq \frac{1}{n-1} \sum_{j \in D} \lambda_{m,j} > 0$$

Proof. Observe that l is not tapping into any signal in D, i.e., $g_{lj} = 0 \, \forall \, j \in D$, thus we have:

$$\sum_{j \notin D} g_{l,j}\beta_{-l,j} = \sum_{j=0}^{n} g_{l,j}\beta_{-l,j} = 1 - \sum_{j=0}^{n}(1 - g_{l,j})\beta_{-l,j}.$$

Player m is tapping into all signals in the set D, i.e., $g_{mj} = 1 \, \forall \, j \in D$, and thus:

$$\sum_{j \notin D} g_{m,j}\beta_{-m,j} = \sum_{j=0}^{n} g_{m,j}\beta_{-m,j} - \sum_{j \in D} \beta_{-m,j} = 1 - \sum_{j=0}^{n}(1 - g_{m,j})\beta_{-m,j} - \sum_{j \in D} \beta_{-m,j}.$$

Subtracting the first from the second, we have:

$$\sum_{j \notin D} g_{m,j}\beta_{-m,j} - \sum_{j \notin D} g_{l,j}\beta_{-l,j} = \sum_{j=0}^{n}(1 - g_{l,j})\beta_{-l,j} - \sum_{j=0}^{n}(1 - g_{m,j})\beta_{-m,j} - \sum_{j \in D} \beta_{-m,j}.$$

We can partition all signals in the economy into four groups. The set of signals they are both observing S_B; the set of signals neither is observing S_N; the set of signals m is observing and l is not, S_M; and the set of signals l is observing and m is not, S_L.

$$\sum_{j \notin D} g_{m,j}\beta_{-m,j} - \sum_{j \notin D} g_{l,j}\beta_{-l,j} = \sum_{j \in S_N} \left(\beta_{-l,j} - \beta_{-m,j}\right) + \sum_{j \in S_M} \beta_{-l,j} - \sum_{j \in S_L} \beta_{-m,j} - \sum_{j \in D} \beta_{-m,j}$$

From Equation (DD.11), $\beta_j = \frac{n-1}{n}\beta_{-m,j} + \frac{1}{n}\lambda_{m,j}$. According to Equation (DD.5) in Appendix A.1.2, we know that $\lambda_{ij} = 0$ whenever $g_{ij} = 0$. Hence, for a signal j in S_N, both $\lambda_{m,j}$ and $\lambda_{l,j}$ are zero. For a signal in S_M, $\lambda_{l,j} = 0$ and for a signal in S_L, $\lambda_{m,j} = 0$. Thus, using these relationships, we can rewrite the difference above as

$$\begin{aligned}
\sum_{j \notin D} g_{m,j}\beta_{-m,j} - \sum_{j \notin D} g_{l,j}\beta_{-l,j} &= \frac{n}{n-1}\left[\sum_{j \in S_M} \beta_j - \sum_{j \in S_L} \beta_j\right] - \sum_{j \in D} \beta_{-m,j} \\
&= \frac{n}{n-1}\left[\underbrace{\sum_{j \in S_M} \beta_j - \sum_{j \in S_L} \beta_j - \sum_{j \in D} \beta_j}_{\geq 0} + \sum_{j \in D} \frac{\lambda_{m,j}}{n}\right] \\
&\geq \frac{1}{n-1} \underbrace{\sum_{j \in D} \lambda_{m,j}}_{>0} > 0
\end{aligned}$$

The final argument is to note that the set of signals player m observes and l does not, S_M, weakly dominates the union of the set D with the set of signals that l observes while m does not, S_L. Remember that $D \subset S_M$, and either $S_L = \{\emptyset\}$ or $S_L = \{e_L\}$. Hence, by definition of the sets S_M, S_L, and D, we have that

$$\sum_{j \in S_M} \beta_j - \sum_{j \in S_L} \beta_j - \sum_{j \in D} \beta_j \geq 0.$$

This is a direct result of agent $m's$ free signal being ranked higher than agent l's signal. Thus, the set of signals m observes always includes signal m and the set of signals agent l observes always includes signal l. □

Using the lemmata proven above, we now proceed to look at agent's payoffs if they deviated and chose to observe different signals. Let $\Delta\Pi_m$ be the difference of ex-ante expected payoff of player m between breaking those d extra links in D or maintaining them. Notice that if player m unilaterally deviates and breaks those links, no other player changes her action. Thus the influence of signals to other players action will not change, and $\beta_{-m,j}$ must be constant. Similarly, let $\Delta\Pi_l$ be the difference of ex-ante expected payoff of player l between observing those d signals or not forming those links. Given that by assumption we are at strict equilibrium, both expected payoff differences should be strictly smaller than zero.

Regarding notation, we keep $g_{i,j}$ to be the original linking strategy, of the proposed equilibrium. Using the payoff function from Proposition 2, we compute $\Delta\Pi_m$ as follows:

$$\Delta\Pi_m = \underbrace{-\frac{\sigma^2\left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)^2}{\sigma^2 + \mathcal{K}_m - d + 1}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{mj}\beta_{-m,j}\right)^2}{\sigma^2 + \mathcal{K}_m + 1}}_{\text{Observing D}}$$
$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{mj})\beta_{-m,j}^2 + \sum_{j\in D}\beta_{-m,j}^2\right)}_{\text{Without observing}}\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{mj})\beta_{-m,j}^2}_{\text{Observing D}} + \underbrace{\Delta C}_{\substack{\text{Cost difference}\\ \text{of not observing signals}}}$$

Breaking up

$$\left(1 - r\sum_{j=0}^{n} g_{mj}\beta_{-m,j}\right)^2 = \left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)^2 - 2r\left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)\sum_{j\in D}\beta_{-m,j} + r^2\left(\sum_{j\in D}\beta_{-m,j}\right)^2,$$

we have:

$$\begin{aligned}\Delta\Pi_m = & -\left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)^2\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_m-d+1} - \frac{\sigma^2}{\sigma^2+\mathcal{K}_m+1}\right)\\ & - 2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_m+1}\left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)\sum_{j\in D}\beta_{-m,j}\\ & + \frac{r^2\sigma^2}{\sigma^2+\mathcal{K}_m+1}\left(\sum_{j\in D}\beta_{-m,j}\right)^2 - r^2\sigma^2\sum_{j\in D}\beta_{-m,j}^2 + \Delta C\end{aligned}$$

If we analyze the ex-ante expected payoff difference for player l, we have:

$$\Delta\Pi_l = \underbrace{-\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{lj}\beta_{-l,j} - r\sum_{j\in D}\beta_{-l,j}\right)^2}{\sigma^2 + \mathcal{K}_l + d + 1}}_{\text{Observing D}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{lj}\beta_{-l,j}\right)^2}{\sigma^2 + \mathcal{K}_l + 1}}_{\text{Without observing}}$$
$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{lj})\beta_{-l,j}^2 - \sum_{j\in D}\beta_{-l,j}^2\right)}_{\text{Observing D}}\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{lj})\beta_{-l,j}^2}_{\text{Without observing}} \underbrace{-\Delta C}_{\substack{\text{Cost difference}\\ \text{of observing signals}}}$$

Notice that $\sum_{j\notin D} g_{lj}\beta_{-lj} = \sum_{j=0}^{n} g_{lj}\beta_{-lj}$ because $g_{lj} = 0\,\forall\, j \in D$. Hence, payoff difference can be written as:

$$\Delta\Pi_l = \left(1 - r\sum_{j\notin D} g_{lj}\beta_{-l,j}\right)^2 \left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_l+1} - \frac{\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\right) + 2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\left(1 - r\sum_{j\notin D} g_{lj}\beta_{-l,j}\right)\sum_{j\in D}\beta_{-l,j}$$
$$- \frac{r^2\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\left(\sum_{j\in D}\beta_{-l,j}\right)^2 + r^2\sigma^2\sum_{j\in D}\beta_{-l,j}^2 + \Delta C$$

Before we proceed, note that the cost difference is the same in both cases, and also that $\mathcal{K}_l + d = \mathcal{K}_m$. Thus, as we sum the two differences, we have:

$$\begin{aligned}\Delta\Pi_l + \Delta\Pi_m = &\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_l+1} - \frac{\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\right)\left[\left(1 - r\sum_{j\notin D} g_{lj}\beta_{-l,j}\right)^2 - \left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)^2\right] \\ &+ 2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\left[\left(1 - r\sum_{j\notin D} g_{lj}\beta_{-l,j}\right)\sum_{j\in D}\beta_{-l,j} - \left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)\sum_{j\in D}\beta_{-m,j}\right] \\ &- \frac{r^2\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\left[\left(\sum_{j\in D}\beta_{-l,j}\right)^2 - \left(\sum_{j\in D}\beta_{-m,j}\right)^2\right] + r^2\sigma^2\left[\sum_{j\in D}\beta_{-l,j}^2 - \sum_{j\in D}\beta_{-m,j}^2\right]\end{aligned}$$

Reordering terms

$$\begin{aligned}\Delta\Pi_l + \Delta\Pi_m = &\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_l+1} - \frac{\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\right)\left[\left(1 - r\sum_{j\notin D} g_{lj}\beta_{-l,j}\right)^2 - \left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)^2\right] \\ &+ 2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_l+d+1}\left[\left(1 - r\sum_{j\notin D} g_{lj}\beta_{-l,j}\right)\sum_{j\in D}\beta_{-l,j} - \left(1 - r\sum_{j\notin D} g_{mj}\beta_{-m,j}\right)\sum_{j\in D}\beta_{-m,j}\right] \\ &+ r^2\sigma^2\left[\left(\sum_{j\in D}\beta_{-l,j}^2 - \sum_{j\in D}\beta_{-m,j}^2\right) - \frac{\left(\sum_{j\in D}\beta_{-l,j}\right)^2 - \left(\sum_{j\in D}\beta_{-m,j}\right)^2}{\sigma^2+K_l+d+1}\right]\end{aligned}$$

We are now ready to sign the first two lines of the expression above. First, $d > 0$ and Lemma A.8 give us that the first line is the product of two strictly positive terms. For the second line, by Lemma A.7 and Lemma A.8, we have that the second line is also strictly positive. In what follows, we show that the third line is non-negative, characterizing the contradiction. To sign the third line is equivalent to sign the following expression:

$$\begin{aligned}&\frac{\left(\sum_{j\in D}\beta_{-l,j}^2 - \sum_{j\in D}\beta_{-m,j}^2\right)}{d} - \frac{d}{\sigma^2+K_l+d+1}\frac{\left(\sum_{j\in D}\beta_{-l,j}\right)^2 - \left(\sum_{j\in D}\beta_{-m,j}\right)^2}{d^2} \\ &\qquad\geq \frac{\sum_{j\in D}\left(\beta_{-l,j}^2 - \beta_{-m,j}^2\right)}{d} - \frac{\left(\sum_{j\in D}\beta_{-l,j}\right)^2 - \left(\sum_{j\in D}\beta_{-m,j}\right)^2}{d^2} \\ &\qquad= \frac{1}{d}\sum_{j\in D}\left(\beta_{-l,j} - \frac{1}{d}\sum_{s\in D}\beta_{-l,s}\right)^2 - \frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j} - \frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)^2\end{aligned}$$

From Equation (DD.11) along with the fact that $\lambda_{lj} = 0$, we know that $\beta_{-l,j} = \beta_{-m,j} + \frac{1}{n-1}\lambda_{m,j}$, and therefore the expression simplifies to:

$$
\begin{aligned}
=&\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j}+\frac{1}{n-1}\lambda_{m,j}-\frac{1}{d}\sum_{s\in D}\left(\beta_{-m,s}+\frac{1}{n-1}\lambda_{m,s}\right)\right)^2-\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j}-\frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)^2\\
=&\left(\frac{1}{n-1}\right)^2\frac{1}{d}\sum_{j\in D}\left(\lambda_{m,j}-\frac{1}{d}\sum_{s\in D}\lambda_{m,s}\right)^2+\left(\frac{2}{n-1}\right)\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j}-\frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)\left(\lambda_{m,j}-\frac{1}{d}\sum_{s\in D}\lambda_{m,s}\right)\\
\geq&\left(\frac{2}{n-1}\right)\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j}-\frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)\left(\lambda_{m,j}-\frac{1}{d}\sum_{s\in D}\lambda_{m,s}\right).
\end{aligned}
$$

We still have to show that this term is positive. It amounts to show that a player considers more, and thus gives it a higher weight (higher $\lambda_{-m,j}$) to more influential signals (higher $\beta_{-m,j}$). The next lemma rewrite λ_{ij} as a function of $\left\{\beta_{-i,j}\right\}_{j=0}^{n}$, and then Lemma A.10 shows that this relation holds.

Lemma A.9. *For every agent i, λ_{ij} can be expressed as a function of $\beta_{-i,j}$ as follows:*

$$\lambda_{i,j}=g_{i,j}\left[\frac{1-r_i\beta_{-i,0}}{\sigma^2+\mathcal{K}_i+1}+r_i\beta_{-i,j}-\frac{r_i}{\sigma^2+\mathcal{K}_i+1}\sum_{s=1}^{n}\beta_{-i,s}g_{i,s}\right]\text{ for } j=1,\ldots,n. \tag{DD.23}$$

Proof. We start by computing the best action, as a function of signals observed. From Appendix A.2, Equations (DD.16) and (DD.17), we know that $a_i^*=F_i'\omega$, and that

$$a_i=\mathbb{E}\left[F_i'\omega|\mathbb{I}_i\right]=\underbrace{F_i'\text{Cov}\left(\omega,X_i\Gamma\omega\right)'\,\text{Var}\left(X_i\Gamma\omega\right)^{-1}X_i}_{[\lambda_{i1},\lambda_{i2},\ldots,\lambda_{in}]}\underbrace{\Gamma\omega}_{[e_1,e_2,\ldots,e_n]'},$$

where F_i, X_i, Γ, and ω were all defined in Appendix A.2, Equations (DD.14), (DD.15), and (DD.16).

Let us start by computing the following term:

$$\text{Cov}\left(\omega,X_i\Gamma\omega\right)'\,\text{Var}\left(X_i\Gamma\omega\right)^{-1}X_i=\begin{bmatrix}B_1\\B_2\end{bmatrix}_{(n+1)\times n},$$

where B_1 is a $1\times n$ matrix given by:

$$B_1=\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i-\frac{1}{\phi_i}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i$$

and B_2 is a $n\times n$ matrix given by:

$$B_2=\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}X_i-\frac{1}{\phi_i}\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i.$$

The matrix Φ is a diagonal matrix of $\{\sigma_i\}_{i=1}^n$, and $\phi_i=1+\sum_{j=1}^n g_{ij}\sigma_j^{-2}$, both terms are defined in Appendix A.2, Equations (DD.20) and (DD.21).

We can use the fact that agents are ex-ante identical,i.e., $\sigma_i=\sigma$ and $r_i=r$ for all $i=1,\ldots,n$, and simplify some expressions as follows:

$$
\begin{aligned}
X_i\Phi\Phi'X_i'&=\sigma^2\mathbf{I}_{\mathcal{K}_i+1}\\
\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i&=\sigma^{-2}[g_{i,1},g_{i,2},\ldots g_{i,n}]_{1\times n}
\end{aligned}
$$

$$\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1} = \sigma^{-2}(\mathcal{K}_i + 1)_{1\times 1}$$
$$\phi_i = 1 + (\mathcal{K}_i + 1)\sigma^{-2}$$

Which gives us a simplified expression for B_1:

$$B_1 = \frac{1}{\sigma^2 + \mathcal{K}_i + 1}[g_{i,1}, g_{i,2}, ...g_{i,n}]_{1\times n}.$$

Similarly, we simply the elements of B_2:

$$\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}X_i = \sigma^{-1}X_i'X_i = \sigma^{-1}\text{diag}([g_{i,1}, g_{i,2}, ...g_{i,n}])_{n\times n}$$
$$\Phi'X_i'(X_i\Phi\Phi'X_i')^{-1}\mathbf{1}\mathbf{1}'(X_i\Phi\Phi'X_i')^{-1}X_i = \sigma^{-3}\begin{bmatrix} g_{i,1} \\ g_{i,2} \\ ... \\ g_{i,n} \end{bmatrix}_{n\times 1}[g_{i,1}, g_{i,2}, ...g_{i,n}]_{1\times n}$$

Thus, B_2 can be written as:

$$B_2 = \sigma^{-1}\text{diag}([g_{i,1}, g_{i,2}, ...g_{i,n}]) - \frac{\sigma^{-1}}{\sigma^2 + \mathcal{K}_i + 1}\begin{bmatrix} g_{i,1} \\ g_{i,2} \\ ... \\ g_{i,n} \end{bmatrix}_{n\times 1}[g_{i,1}, g_{i,2}, ...g_{i,n}]$$

Next, we combine the simplified expressions for B_1 and B_2 along with the definition of F_i. For any signal e_j, $j \in \{1, 2, ..., n\}$, we can compute the linear coefficient λ_{ij} of that particular signal over the action of agent i:

$$[\lambda_{i1}, \lambda_{i2}, ..., \lambda_{in}] = F_i'\begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $F_i' = [1 - r_i\beta_{-i,0},\ r_i\sigma\beta_{-i,1},\ r_i\sigma\beta_{-i,2},\ \cdots,\ r_i\sigma\beta_{-i,n}]$. Hence:

$$\begin{aligned}\lambda_{i,j} &= \frac{1 - r_i\beta_{-i,0}}{\sigma^2 + \mathcal{K}_i + 1}g_{i,j} + r_i\beta_{-i,j}g_{i,j} - \frac{r_i}{\sigma^2 + \mathcal{K}_i + 1}g_{i,j}\sum_{s=1}^{n}\beta_{-i,s}g_{i,s} \\ &= g_{i,j}\left[\frac{1 - r_i\beta_{-i,0}}{\sigma^2 + \mathcal{K}_i + 1} + r_i\beta_{-i,j} - \frac{r_i}{\sigma^2 + \mathcal{K}_i + 1}\sum_{s=1}^{n}\beta_{-i,s}g_{i,s}\right]\end{aligned}$$

□

Lemma A.10. *Consider a subset D of an agent m's information set, i.e., $g_{mj} = 1$ for every $j \in D$. The covariance between the influence a signal has over the average action not including agent m's action and how influential that particular signal is to agent m's action is non-negative.*

$$\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j} - \frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)\left(\lambda_{m,j} - \frac{1}{d}\sum_{s\in D}\lambda_{m,s}\right) \geq 0$$

Proof. Using Lemma A.9, when agents are ex-ante identical we have that:

$$\lambda_{i,j} = g_{i,j}\left[\frac{1 - r\beta_{-i,0}}{\sigma^2 + \mathcal{K}_i + 1} + r\beta_{-i,j} - \frac{r}{\sigma^2 + \mathcal{K}_i + 1}\sum_{s=1}^{n}\beta_{-i,s}g_{i,s}\right].$$

For $i = m$ and $j \in D$, we have that $g_{mj} = 1$. Hence,

$$\lambda_{m,j} = \frac{1 - r\beta_{-m,0}}{\sigma^2 + \mathcal{K}_m + 1} + r\beta_{-m,j} - \frac{r}{\sigma^2 + \mathcal{K}_m + 1}\sum_{s=1}^{n}\beta_{-m,s}g_{m,s}$$

for every $j \in D$.

Now that we have written $\lambda_{m,j}$ for $j \in D$ as a function of $\{\beta_{-i,s}\}_{s=1}$, we conclude the proof of this lemma as follows:

$$\begin{aligned}
&\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j} - \frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)\left(\lambda_{m,j} - \frac{1}{d}\sum_{s\in D}\lambda_{m,s}\right)\\
&= \frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j} - \frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)\left(r\beta_{-m,j} - \frac{1}{d}\sum_{s\in D}r\beta_{-m,s}\right)\\
&= r\frac{1}{d}\sum_{j\in D}\left(\beta_{-m,j} - \frac{1}{d}\sum_{s\in D}\beta_{-m,s}\right)^2 \geq 0.
\end{aligned}$$

□

This gives us that $\Delta\Pi_m + \Delta\Pi_l > 0$, even though by assumption both elements were smaller than zero, characterizing our contradiction.

B.2 Example of out-of-equilibrium violation of Proprerties 1 and 2

In this section, we provide a numerical example highlighting that Properties 1 and 2 do not hold in out of equilibrium networks. Let us assume an economy with 10 agents with identical preference parameters given by $r = 0.5$ and $\sigma^2 = 1$. Also, we assume a linear cost function given by $c(\mathcal{K}) = 0.12\mathcal{K}$.

Furthermore, agents are connected as follows:

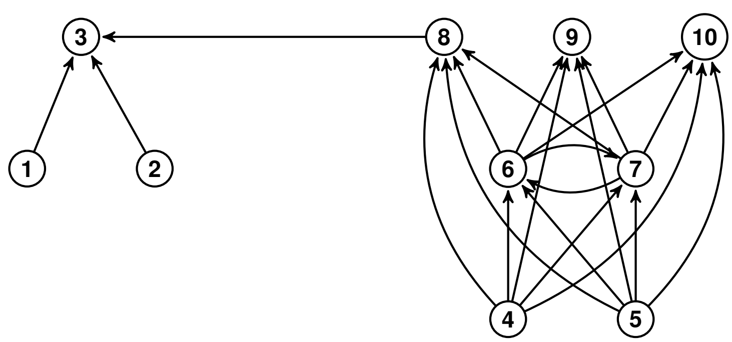


Figure 2: Example showing that Properties 1 and 2 do not hold in out of equilibrium networks.

In this informational structure, agents 1, 2, and 8 observe agent 3's signal in additional to their own respective signal and the common prior. Agents 3, 9, and 10 do not observe any additional signal. Finally, agents 4, 5, 6, and 7 observe agent 6's, 7's, 8's, 9's, and 10's signal in addition to their own respective signal and the common prior.

Although this network is not an equilibrium, we can compare all possible deviations using the payoff deviation formula from Proposition 2, and numerically verify that agents 3 and 8 do not want to deviate, taking all other agents actions and connections as given. That is, all agents choose optimal actions given their connections and agents 3 and 8 do not want to form or break connections.

This example violates Property 1 because agent 8 wants to keep observing agent 3 instead of, for instance, agent 10. However, agent 10's signal is observed by 4 other agents in addition to agent 10 herself, i.e. $\overline{\mathcal{K}}_{10} = 4$, while 3's signal is observed by 3 other agents in addition to agent 3 herself, i.e. $\overline{\mathcal{K}}_3 = 3$. Hence, $\mathcal{K}_{10} > \mathcal{K}_3$. This is a clear violation of Property 1.

Property 2 does not hold in this example because agent 3 wants to keep observing no additional signal, i.e., $\mathcal{K}_3 = 0$, while agent 8 wants to keep observing agent 3's signal, i.e. $\mathcal{K}_8 = 1$. However, agent 8's signal is observed by 4 other agents in additional to agent 8 herself, i.e. $\overline{\mathcal{K}}_8 = 4$, while 3's signal is observed by 3 other agents in additional to agent 3 herself, i.e. $\overline{\mathcal{K}}_3 = 3$. This violates Property 2 since $\overline{\mathcal{K}}_8 > \overline{\mathcal{K}}_3$ and $\mathcal{K}_8 > \mathcal{K}_3$.

B.3 Proof of Proposition 4

Given Assumption 1, Properties 1 and 2 hold in equilibrium (Proposition 3), and any strict Nash equilibrium is a hierarchical directed network (Theorem 1 and Corollary 1). Hence, we have to show that Property 3 holds in equilibrium given assumptions 1 and 2.

We will prove Proposition 4 by contradiction. Let us assume that there are three agents, a, b, and c, such that c is connected to both a and b, i.e., $g_{ca} = 1$ and $g_{cb} = 1$, but a and b are not connected to each other, i.e., $g_{ab} = 0$. We know that the equilibrium features a hierarchical network as information structure. Hence, the connections between these three agents characterize different hierarchical networks, and there are only three possibilities:[5] (i) networks in which members of the top tier do not observe each other (a and b in the top tier); and (ii) networks in which members of the bottom tier observe each other (b and c in the bottom tier); or (iii) networks with more than two tiers (a, b, and c each in a different tier).

Due to the hierarchical structure, agent c observes at least as many signals as agent a in addition to the signal from b and her own. Thus, if agent c stopped observing agent b's signal, her information set would still be a strict super set of agent a's.

We will compare two deviations, for agent c to break the link with all players she is observing and player a is not, call such set D, and for player a to form links with such set. Note that the set D is not empty, and has $d \geq 1$ elements. Note also that, even after player a forms links with all players in D, player a's information set will still be a strict subset of player c's. The reason is that c is observing a, but the converse is not true. This follows from the definition of hierarchical network. If a and b are in same tier (case i above), than it must be that $g_{ac} = 0$ because $g_{ab} = 0$. If a and b are not in the same tier, there are two possibilities. One possibility is that b and c are in the same tier (case ii above), then $g_{cb} = 1$ implies $g_{bc} = 1$ (full tier). Also, $g_{ca} = 1$ implies $g_{ba} = 1$ and thus a is in a tier above b and c's tier; and, since $g_{ab} = 0$ and the tier of agents b and c is full, then $g_{ac} = 0$ because otherwise c would be in a's tier and a would have to observe b's signal as well. Finally, the other possibility is that a, b, and c are in different tiers (case iii above), then we know that b is in a tier above c's, and thus $g_{ab} = 0$ implies $g_{ac} = 0$.

Notice that

$$\mathcal{K}_a + d = \mathcal{K}_c - 1 < \mathcal{K}_c$$

[5]Notice that these are the only three possibilities: (i) $g_{ba} = 0$; (ii) $g_{ba} = 1$ and $g_{bc} = 1$; or (iii) $g_{ba} = 1$ and $g_{bc} = 0$.

because c is connected to a but a is not connected to c but , which makes $d = \mathcal{K}_c - \mathcal{K}_a - 1$. First we use Proposition 2 to present the payoff difference for player c, comparing the payoff of deviating with the payoff of maintaining the links.

$$\Delta\Pi_c = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D} g_{cj}\beta_{-c,j}\right)^2}{\sigma^2 + \mathcal{K}_c - d + 1}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{cj}\beta_{-c,j}\right)^2}{\sigma^2 + \mathcal{K}_c + 1}}_{\text{Observing D}}$$

$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{cj})\beta^2_{-c,j} + \sum_{j\in D}\beta^2_{-c,j}\right)}_{\text{Without observing}} \underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{cj})\beta^2_{-c,j}}_{\text{Observing D}} + \underbrace{C(\mathcal{K}_c) - C(\mathcal{K}_c - d)}_{\text{Cost difference of not observing signals}}$$

$$\begin{aligned}
\Delta\Pi_c =& -\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)^2\left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_c - d + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\right) - r^2\sigma^2\sum_{j\in D}\beta^2_{-c,j} \\
&- 2\frac{r\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)\sum_{j\in D}\beta_{-c,j} + \frac{r^2\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\left(\sum_{j\in D}\beta_{-c,j}\right)^2 + C(\mathcal{K}_c) - C(\mathcal{K}_c - d) \\
\Delta\Pi_c =& -\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)^2\left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_c - d + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\right) \\
&- 2\frac{r\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)\sum_{j\in D}\beta_{-c,j} \\
&- r^2\sigma^2\left[\sum_{j\in D}\beta^2_{-c,j} - \frac{1}{\sigma^2 + \mathcal{K}_c + 1}\left(\sum_{j\in D}\beta_{-c,j}\right)^2\right] + C(\mathcal{K}_c) - C(\mathcal{K}_c - d)
\end{aligned}$$

For this not be be a profitable deviation for agent c, it must be that $\Pi_c < 0$, which implies that:

$$\begin{aligned}
C(\mathcal{K}_c) - C(\mathcal{K}_c - d) <& \left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)^2\left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_c - d + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\right) \\
&+ 2\frac{r\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)\sum_{j\in D}\beta_{-c,j} \\
&+ r^2\sigma^2\left[\sum_{j\in D}\beta^2_{-c,j} - \frac{1}{\sigma^2 + \mathcal{K}_c + 1}\left(\sum_{j\in D}\beta_{-c,j}\right)^2\right]
\end{aligned}$$

Following similar steps for agent a, we compute a's payoff gain, i.e., $\Delta\Pi_a$, from deviating and observing the signals from D. We have that:

$$\Delta\Pi_a = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j} - r\sum_{j\in D}\beta_{-a,j}\right)^2}{\sigma^2 + \mathcal{K}_a + d + 1}}_{\text{Observing D}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{a,j}\beta_{-a,j}\right)^2}{\sigma^2 + \mathcal{K}_a + 1}}_{\text{Without observing}}$$

$$
\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{aj})\beta_{-a,j}^2-\sum_{j\in D}\beta_{-a,j}^2\right)}_{\text{Observing D}}+\underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{aj})\beta_{-a,j}^2\right)}_{\text{Without observing}}-\underbrace{[C(\mathcal{K}_a+d)-C(\mathcal{K}_a)]}_{\text{Cost difference}}
$$

$$
\begin{aligned}
\Delta\Pi_a = & -\left(1-r\sum_{j\notin D}g_{a,j}\beta_{-a,j}\right)^2\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}-\frac{\sigma^2}{\sigma^2+\mathcal{K}_a+1}\right)\\
& +2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}\left(1-r\sum_{j\notin D}g_{a,j}\beta_{-a,j}\right)\sum_{j\in D}\beta_{-a,j}\\
& -\frac{r^2\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}\left(\sum_{j\in D}\beta_{-a,j}\right)^2+r^2\sigma^2\sum_{j\in D}\beta_{-a,j}^2-[C(K_a+d)-C(K_a)]
\end{aligned}
$$

$$
\begin{aligned}
\Delta\Pi_a = & -\left(1-r\sum_{j\notin D}g_{a,j}\beta_{-a,j}\right)^2\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}-\frac{\sigma^2}{\sigma^2+\mathcal{K}_a+1}\right)\\
& +2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}\left(1-r\sum_{j\notin D}g_{a,j}\beta_{-a,j}\right)\sum_{j\in D}\beta_{-a,j}\\
& +r^2\sigma^2\left[\sum_{j\in D}\beta_{-a,j}^2-\frac{1}{\sigma^2+\mathcal{K}_a+d+1}\left(\sum_{j\in D}\beta_{-a,j}\right)^2\right]-[C(\mathcal{K}_a+d)-C(\mathcal{K}_a)]
\end{aligned}
$$

For this not be be a profitable deviation for agent a, it must be that $\Pi_a < 0$, which implies that:

$$
\begin{aligned}
C(\mathcal{K}_a+d)-C(\mathcal{K}_a) > & \left(1-r\sum_{j\notin D}g_{a,j}\beta_{-a,j}\right)^2\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_a+1}-\frac{\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}\right)\\
& +2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_a+d+1}\left(1-r\sum_{j\notin D}g_{a,j}\beta_{-a,j}\right)\sum_{j\in D}\beta_{-a,j}\\
& +r^2\sigma^2\left[\sum_{j\in D}\beta_{-a,j}^2-\frac{1}{\sigma^2+\mathcal{K}_a+d+1}\left(\sum_{j\in D}\beta_{-a,j}\right)^2\right]
\end{aligned}
$$

Given that $\mathcal{K}_c = K_a + d + 1 > K_a + d$, by convexity of the cost curve (Assumption 2), we have that:[6]

$$
C(\mathcal{K}_c)-C(\mathcal{K}_c-d)\geq C(\mathcal{K}_a+d)-C(\mathcal{K}_a).
$$

This inequality disciplines how the expressions we have just obtained relate.

$$
\begin{aligned}
& \left(1-r\sum_{j\notin D}g_{c,j}\beta_{-c,j}\right)^2\left(\frac{\sigma^2}{\sigma^2+\mathcal{K}_c-d+1}-\frac{\sigma^2}{\sigma^2+\mathcal{K}_c+1}\right)\\
& +2\frac{r\sigma^2}{\sigma^2+\mathcal{K}_c+1}\left(1-r\sum_{j\notin D}g_{c,j}\beta_{-c,j}\right)\sum_{j\in D}\beta_{-c,j}+r^2\sigma^2\left[\sum_{j\in D}\beta_{-c,j}^2-\frac{1}{\sigma^2+\mathcal{K}_c+1}\left(\sum_{j\in D}\beta_{-c,j}\right)^2\right]\\
& >\\
& C(\mathcal{K}_c)-C(\mathcal{K}_c-d)
\end{aligned}
$$

[6]In the assumption 2 notation, we would have: $X = \mathcal{K}_c - 1$ and $Y = \mathcal{K}_a$.

$$\geq$$

$$C(\mathcal{K}_a + d) - C(\mathcal{K}_a)$$

$$>$$

$$\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right)^2 \left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_a + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_a + d + 1}\right)$$

$$+ 2\frac{r\sigma^2}{\sigma^2 + \mathcal{K}_a + d + 1}\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right)\sum_{j\in D}\beta_{-a,j} + r^2\sigma^2\left[\sum_{j\in D}\beta_{-a,j}^2 - \frac{1}{\sigma^2 + \mathcal{K}_a + d + 1}\left(\sum_{j\in D}\beta_{-a,j}\right)^2\right]$$

This gives us the following inequality.

$$\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)^2 \left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_c - d + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\right) + r^2\sigma^2\sum_{j\in D}\beta_{-c,j}^2$$

$$\frac{r\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\left(\sum_{j\in D}\beta_{-c,j}\right)\left[2\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right) - r\sum_{j\in D}\beta_{-c,j}\right]$$

$$>$$

$$\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right)^2 \left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_a + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_a + d + 1}\right) + r^2\sigma^2\sum_{j\in D}\beta_{-a,j}^2$$

$$+ \frac{r\sigma^2}{\sigma^2 + \mathcal{K}_a + d + 1}\left(\sum_{j\in D}\beta_{-a,j}\right)\left[2\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right) - r\sum_{j\in D}\beta_{-a,j}\right]$$

We will show that the above inequality never holds, characterizing a contradiction. Observe that the following inequalities hold: (i) $\mathcal{K}_c - d > \mathcal{K}_a$ because $d = \mathcal{K}_c - \mathcal{K}_a - 1$; (ii) $\beta_{-a,j} > \beta_{-c,j}$ for every $j \in D$ from Lemma A.7; and (iii) $\sum_{j\notin D} g_{c,j}\beta_{-c,j} \geq \sum_{j\notin D} g_{a,j}\beta_{-a,j}$ from Lemma A.8.[7] These three inequalities combined imply:

$$\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right)^2 \left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_c - d + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\right) < \left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right)^2 \left(\frac{\sigma^2}{\sigma^2 + \mathcal{K}_a + 1} - \frac{\sigma^2}{\sigma^2 + \mathcal{K}_a + d + 1}\right)$$

and

$$r^2\sigma^2\sum_{j\in D}\beta_{-c,j}^2 < r^2\sigma^2\sum_{j\in D}\beta_{-a,j}^2$$

Finally, we can work with the last term so that:

$$\frac{r\sigma^2}{\sigma^2 + \mathcal{K}_c + 1}\left(\sum_{j\in D}\beta_{-c,j}\right)\left[2\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right) - r\sum_{j\in D}\beta_{-c,j}\right] < \frac{r\sigma^2}{\sigma^2 + \mathcal{K}_a + d + 1}\left(\sum_{j\in D}\beta_{-a,j}\right)\left[2\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right) - r\sum_{j\in D}\beta_{-a,j}\right]$$

because (i) $\mathcal{K}_c > \mathcal{K}_a + d$, which makes $\frac{r\sigma^2}{\sigma^2+\mathcal{K}_a+d+1} > \frac{r\sigma^2}{\sigma^2+\mathcal{K}_c+1}$; (ii) from Lemma A.7 we have that $\sum_{j\in D}\beta_{-a,j} > \sum_{j\in D}\beta_{-c,j}$; and (iii) using Lemma A.8 along with the definitions of $\beta_{-a,j}$ and $\beta_{-c,j}$ (Equation DD.11) and the fact that $\lambda_{aj} = 0$ for

[7] The condition in Lemma A.8 holds with equality by the definition of the set D.

every $j \in D$, we have that

$$
\begin{aligned}
&2\left(1 - r\sum_{j\notin D} g_{a,j}\beta_{-a,j}\right) - r\sum_{j\in D}\beta_{-a,j} - \left[2\left(1 - r\sum_{j\notin D} g_{c,j}\beta_{-c,j}\right) - r\sum_{j\in D}\beta_{-c,j}\right] \\
&= -2r\sum_{j\notin D} g_{a,j}\beta_{-a,j} - r\sum_{j\in D}\beta_{-a,j} + 2r\sum_{j\notin D} g_{c,j}\beta_{-c,j} + r\sum_{j\in D}\beta_{-c,j} \\
&= 2r\left(\sum_{j\notin D} g_{c,j}\beta_{-c,j} - \sum_{j\notin D} g_{a,j}\beta_{-a,j}\right) - r\sum_{j\in D}\beta_{-a,j} + r\sum_{j\in D}\beta_{-c,j} \\
&\geq 2r\left(\frac{1}{n-1}\sum_{j\in D}\lambda_{c,j}\right) - r\sum_{j\in D}\beta_{-a,j} + r\sum_{j\in D}\beta_{-c,j} \\
&= r\left(2\frac{1}{n-1}\sum_{j\in D}\lambda_{c,j} - \frac{1}{n-1}\sum_{j\in D}\lambda_{c,j}\right) \\
&= r\frac{1}{n-1}\sum_{j\in D}\lambda_{c,j} \\
&> 0.
\end{aligned}
$$

This finally characterizes our contradiction.

B.4 Three-Tier Equilibrium

Let us consider an economy with 5 agents, $\sigma = 1$, and $r = 0.5$. We assume the following non-convex cost function:

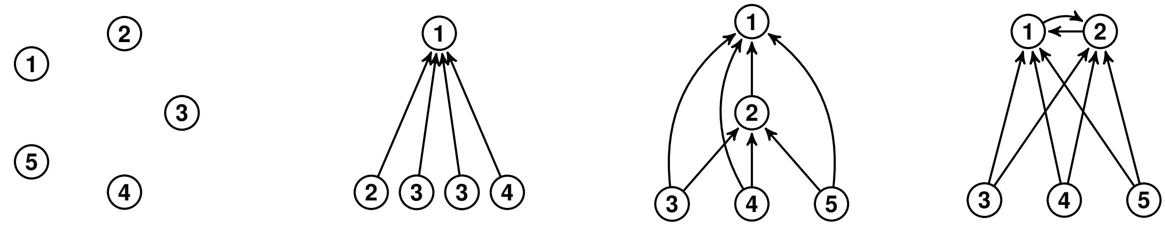
$$
c(\mathcal{K}_i) = \begin{cases} 0 & \text{if } \mathcal{K}_i = 0 \\ 0.14 & \text{if } \mathcal{K}_i = 1 \\ 0.20 & \text{if } \mathcal{K}_i = 2 \\ \infty & \text{if } \mathcal{K}_i \geq 3. \end{cases}
$$

For this set of parameters, there are four different equilibrium information structures, represented in Figure 3. In order to highlight the role of Assumption 2 in Theorem 2, this example violates such an assumption, and we show the existence of an equilibrium that does not have a core-periphery structure. We have in Panels (a), (b), and (d) of Figure 3 different core-periphery networks, while in Panel (c), we have a hierarchical directed network that is not core-periphery. The network in Panel (c) is a three-tier hierarchical network and is an equilibrium.

In the three-tier hierarchical network, agent 1 is the first-tier, agent 2 is the second-tier and agents 3, 4, and 5 are the third-tier. This holds in equilibrium because the cost function is non-convex in such a way that the additional cost of observing two agents' signals instead of only one is relatively low. Hence, third-tier agents are willing to observe two additional signals, one from agent 1 and another from agent 2. Agent 2's signal is more informative about the average action than are signals from the third tier. This wedge between signals' informativeness makes agent 2 (second tier) willing to observe only the signal from the first tier, since she already observes her own signal. Assumption 2 prevents a scenario like this one from happening. If assumption 2 holds, then agents in the third tier would find it expensive to observe agent 2's signal, and a three-tier hierarchical network would not be an equilibrium.

Figure 3: Example of equilibrium with a three-tier hierarchical network

This picture shows the equilibrium networks of example in Section B.4. We numerically solve for the equilibria, and there are five equilibria with different information structures. All five equilibrium networks are displayed in the five panels below.



(a) Empty (b) Two tiers (c) Three tiers (d) Two tiers

C Existence of Equilibrium—Proof of Theorem 3

In this section we prove the existence of equilibrium if the cost of obtaining information has weakly increasing differences, Theorem 3 from the main text. From our analysis in Proposition 4 and Theorem 2, and thus the result in Corollary 2 in the main text, we know that if a strict equilibrium exists, it must be a core-periphery directed network. Here, we show that for any cost function satisfying Assumption 2, a pure strategy equilibrium characterized by a core-periphery directed network always exists. Furthermore, we show that for any core-periphery network, G, there exists a cost function satisfying Assumption 2, such that G, is a pure strategy equilibrium.

Before we proceed to the proof, let me introduce two useful pieces of notation. The first one regards different types of core-periphery networks, and the second regards the cost of acquiring information.

First, note that there are two types of possible core-periphery networks. In the first type, simple-core-periphery, SCP, agents in the core observe only other core agents' signals. In the second type, core-periphery-observing-down, CPOD, all agents in the core observe the signal of all other agents in the core and also one extra signal from one particular agent on the periphery. A network of the first type or second type with $0 \leq n_c \leq n$ members in the core, will be denoted $SCP(n_c)$ and $CPOD(n_c)$ respectively.

Second, because the cost function is symmetric for all agents, we can focus on the marginal cost of acquiring a signal. Let us denote the cost of acquiring the l^{th} signal $C(l) - C(l-1) = c_l$.

Consider a fixed sequence of weakly increasing marginal costs, $c_1 \geq c_2 \geq ... \geq c_n$, we show that there exists a core-periphery network that is a pure strategy equilibrium. Our proof will be done by contradiction. We assume that no core-periphery network is a pure strategy equilibrium. We first limit the relevant deviations from a core-periphery network, and show that a particular core-periphery network is an equilibrium if and only if two simple conditions hold. Guaranteeing that two particular deviations are not profitable is enough to confirm that a certain network is an equilibrium. We proceed by using induction on the number of elements in the core. First, we show that if the empty network is not an equilibrium, then it must be that the cost of observing one signal is too cheap. Next, we show that if no network with 1 member on the core is an equilibrium (and the cost of observing one signal is too small for the empty network to be an equilibrium), then it must be that the cost of observing two signals is too small. Using the monotonicity of marginal costs, we proceed by induction, showing that if no core-periphery network, with a core size n_c is an equilibrium, then the cost of observing the $n_c + 1^{th}$ signal is too cheap. Given that there exist a finite number of players, we soon reach the conclusion that the complete network must be an equilibrium, as the cost of observing all signals is small enough.

In Section C.1, we show three lemmata with equilibrium properties of SCP and CPOD networks, which are used later in the proof. In Section C.2, we identify all relevant deviations from core-periphery networks. In Section C.3, we provee the existence results. Finally, in Section C.4, we show the second result that for any core-periphery network, G, there exists a cost function satisfying Assumptions 1 and 2 such that G is an strict Nash equilibrium.

C.1 Equilibrium Properties of SCP and CPOD Networks

Lemma A.11. *For any simple-core-periphery network with n_c core players, $SCP(n_c)$, for every $n_c = 1, \ldots, n-1$, we have:*

1. $\beta_p^{SCP(n_c)} < \beta_c^{SCP(n_c)} = \frac{1}{\sigma^2}\beta_0^{SCP(n_c)}$,
2. $\beta_{-c,0}^{SCP(n_c)} = \sigma^2 \beta_{-c,c}^{SCP(n_c)}$,
3. $\beta_{-c,p}^{SCP(n_c)} = \beta_{-p,p}^{SCP(n_c)} < \beta_{-p,c}^{SCP(n_c)} = \frac{1}{\sigma^2}\beta_{-p,0}^{SCP(n_c)}$,
4. $\tilde{r}\left(\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)}\right) \leq \frac{1}{\sigma^2+n_c}$ *and* $\beta_{-p,c}^{SCP(n_c)}\left(\sigma^2 + n_c\right) \leq 1$,
5. $\beta_{-p,0}^{SCP(1)} < \beta_{-p,0}^{SCP(0)}$, *and*
6. $\beta_{-p,p}^{SCP(0)} < \beta_{-p,c}^{SCP(1)}$.

where $\beta_0^{SCP(n_c)}$ is the weight of the common prior on the average action ($\overline{a}$), $\beta_c^{SCP(n_c)}$ is the weight of a core player's signal on $\overline{a}$, $\beta_p^{SCP(n_c)}$ is the weight of a peripheral player's signal on $\overline{a}$, $\beta_{-c,0}^{SCP(n_c)}$ is the weight of the common prior on the average action excluding a core agent' signal ($\overline{a}_{-c}$), $\beta_{-c,c}^{SCP(n_c)}$ is the weight of a core player's signal on $\overline{a}_{-c}$, $\beta_{-c,p}^{SCP(n_c)}$ is the weight of a peripheral player's signal on $\overline{a}_{-c}$, $\beta_{-p,0}^{SCP(n_c)}$ is the weight of the common prior on the average action excluding a peripheral agent' signal ($\overline{a}_{-p}$), $\beta_{-p,c}^{SCP(n_c)}$ is the weight of a core player's signal on $\overline{a}_{-p}$, and $\beta_{-p,p}^{SCP(n_c)}$ is the weight of another peripheral player's signal on $\overline{a}_{-p}$.

Proof. This proof heavily rely on Equations (DD.5), (DD.7), and (DD.10) applied to to simple-core-periphery network with n_c core players.

1. From Equation (DD.7), we have that $SPC(n_c)$ betas satisfy:

$$\left[n - \tilde{r}\left(\overline{\mathcal{K}}_j + 1\right)\right]\beta_j = (1-\tilde{r})\sum_{i=1}^{n}\tilde{g}_{ij} + \tilde{r}\sum_{i=1}^{n}\sum_{s=0}^{n}\beta_s(1-g_{is})\tilde{g}_{ij}.$$

Hence, betas solve the following system of equations:

$$[n - \tilde{r}n]\beta_0^{SCP(n_c)} = (1-\tilde{r})\sigma^2\left[\frac{n_c}{\sigma^2+n_c} + \frac{(n-n_c)}{\sigma^2+n_c+1}\right] + \tilde{r}\sigma^2\left[\frac{n_c(n-n_c)}{\sigma^2+n_c} + \frac{(n-n_c)(n-n_c-1)}{\sigma^2+n_c+1}\right]\beta_p^{SCP(n_c)} \tag{DD.24}$$

$$[n - \tilde{r}n]\beta_c^{SCP(n_c)} = (1-\tilde{r})\left[\frac{n_c}{\sigma^2+n_c} + \frac{(n-n_c)}{\sigma^2+n_c+1}\right] + \tilde{r}\left[\frac{n_c(n-n_c)}{\sigma^2+n_c} + \frac{(n-n_c)(n-n_c-1)}{\sigma^2+n_c+1}\right]\beta_p^{SCP(n_c)} \tag{DD.25}$$

$$[n - \tilde{r}]\beta_p^{SCP(n_c)} = (1-\tilde{r})\frac{1}{\sigma^2+n_c+1} + \tilde{r}\frac{(n-n_c-1)}{\sigma^2+n_c+1}\beta_p^{SCP(n_c)} \tag{DD.26}$$

We can solve Equation (DD.26) for $\beta_p^{SCP(n_c)}$:

$$\beta_p^{SCP(n_c)} = \frac{1-\tilde{r}}{\sigma^2(n-\tilde{r}) + n(n_c+1-\tilde{r})} \tag{DD.27}$$

Equations (DD.24) and (DD.25) imply that

$$\beta_c^{SCP(n_c)}\sigma^2 = \beta_0^{SCP(n_c)}$$

We can rearrange Equation (DD.25 as follows:

$$\begin{aligned}\beta_c^{SCP(n_c)} = \frac{1}{n(1-\tilde{r})}&\left\{\left[(1-\tilde{r})\frac{1}{\sigma^2+n_c+1} + \tilde{r}\frac{(n-n_c-1)}{\sigma^2+n_c+1}\beta_p^{SCP(n_c)}\right]\right.\\ &\left.\times\left(n-n_c+n_c\frac{\sigma^2+n_c+1}{\sigma^2+n_c}\right) + \tilde{r}\frac{n_c}{\sigma^2+n_c}\beta_p^{SCP(n_c)}\right\}\end{aligned}$$

substituting Equation (DD.26):

$$= \frac{1}{n(1-\tilde{r})}\left\{[n-\tilde{r}]\left(n-n_c+n_c\frac{\sigma^2+n_c+1}{\sigma^2+n_c}\right)\beta_p^{SCP(n_c)} + \tilde{r}\frac{n_c}{\sigma^2+n_c}\beta_p^{SCP(n_c)}\right\}$$

which can be further simplified to:

$$\beta_c^{SCP(n_c)} = \beta_p^{SCP(n_c)}\frac{n-\tilde{r}+\frac{n_c}{\sigma^2+n_c}}{1-\tilde{r}} = \frac{n-\tilde{r}+\frac{n_c}{\sigma^2+n_c}}{\sigma^2(n-\tilde{r})+n(n_c+1-\tilde{r})}$$

Therefore:

$$\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)} = \beta_p^{SCP(n_c)}\frac{n-1+\frac{n_c}{\sigma^2+n_c}}{1-\tilde{r}} > 0. \tag{DD.28}$$

2. From Equation (DD.10), we have that:

$$\begin{aligned}(n-1)\beta_{-c,0}^{SCP(n_c)} &= (n_c-1)\lambda_{c0}^{SPC(n_c)} + (n-n_c)\lambda_{p0}^{SPC(n_c)}\\ (n-1)\beta_{-c,c}^{SCP(n_c)} &= (n_c-1)\lambda_{cc}^{SPC(n_c)} + (n-n_c)\lambda_{pc}^{SPC(n_c)}\end{aligned}$$

The λ's from Equation (DD.5) can be written as:

$$\lambda_{ij} = \tilde{g}_{ij} + \tilde{r}\beta_j g_{ij} - \tilde{r}\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\tilde{g}_{ij}$$

Under $SPC(n_c)$, we have

$$\begin{aligned}\lambda_{c0}^{SCP(n_c)} &= \frac{\sigma^2}{\sigma^2+n_c} + \tilde{r}\beta_0^{SCP(n_c)} - \tilde{r}\frac{\sigma^2}{\sigma^2+n_c}\left(\beta_0^{SCP(n_c)} + n_c\beta_c^{SCP(n_c)}\right)\\ \lambda_{cc}^{SCP(n_c)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\beta_c^{SCP(n_c)} - \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_0^{SCP(n_c)} + n_c\beta_c^{SCP(n_c)}\right)\\ \lambda_{p0}^{SCP(n_c)} &= \frac{\sigma^2}{\sigma^2+n_c+1} + \tilde{r}\beta_0^{SCP(n_c)} - \tilde{r}\frac{\sigma^2}{\sigma^2+n_c+1}\left(\beta_0^{SCP(n_c)} + n_c\beta_c^{SCP(n_c)} + \beta_p^{SCP(n_c)}\right)\\ \lambda_{pc}^{SCP(n_c)} &= \frac{1}{\sigma^2+n_c+1} + \tilde{r}\beta_c^{SCP(n_c)} - \tilde{r}\frac{1}{\sigma^2+n_c+1}\left(\beta_0^{SCP(n_c)} + n_c\beta_c^{SCP(n_c)} + \beta_p^{SCP(n_c)}\right)\end{aligned}$$

$$\lambda_{pp}^{SCP(n_c)} = \frac{1}{\sigma^2 + n_c + 1} + \tilde{r}\beta_p^{SCP(n_c)} - \tilde{r}\frac{1}{\sigma^2 + n_c + 1}\left(\beta_0^{SCP(n_c)} + n_c\beta_c^{SCP(n_c)} + \beta_p^{SCP(n_c)}\right)$$

where $\lambda_{pp}^{SCP(n_c)}$ is the weight of a peripheral agent own her own signal.

Using the previous item in this lemma, these λs simply to:

$$\lambda_{c0}^{SCP(n_c)} = \frac{\sigma^2}{\sigma^2 + n_c} \tag{DD.29}$$

$$\lambda_{cc}^{SCP(n_c)} = \frac{1}{\sigma^2 + n_c} \tag{DD.30}$$

$$\lambda_{p0}^{SCP(n_c)} = \frac{\sigma^2}{\sigma^2 + n_c + 1} + \tilde{r}\frac{\sigma^2}{\sigma^2 + n_c + 1}\left(\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)}\right) \tag{DD.31}$$

$$\lambda_{pc}^{SCP(n_c)} = \frac{1}{\sigma^2 + n_c + 1} + \tilde{r}\frac{1}{\sigma^2 + n_c + 1}\left(\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)}\right) \tag{DD.32}$$

$$\lambda_{pp}^{SCP(n_c)} = \frac{1}{\sigma^2 + n_c + 1} - \tilde{r}\frac{\sigma^2 + n_c}{\sigma^2 + n_c + 1}\left(\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)}\right) \tag{DD.33}$$

Notice that $\lambda_{pp}^{SCP(n_c)} < \lambda_{pc}^{SCP(n_c)}$, $\lambda_{p0}^{SCP(n_c)} = \sigma^2\lambda_{pc}^{SCP(n_c)}$, and $\lambda_{c0}^{SCP(n_c)} = \sigma^2\lambda_{cc}^{SCP(n_c)}$.

Substituting these into the expressions for $\beta_{-c,0}^{SCP(n_c)}$ and $\beta_{-c,c}^{SCP(n_c)}$, we have that

$$\beta_{-c,0}^{SCP(n_c)} = \sigma^2\beta_{-c,c}^{SCP(n_c)}$$

3. From Equation (DD.10), we have that:

$$\begin{aligned}
(n-1)\beta_{-c,p}^{SCP(n_c)} &= \lambda_{pp} \\
(n-1)\beta_{-p,p}^{SCP(n_c)} &= \lambda_{pp} \\
(n-1)\beta_{-p,c}^{SCP(n_c)} &= (n_c)\lambda_{cc} + (n - n_c - 1)\lambda_{pc} \\
(n-1)\beta_{-p,0}^{SCP(n_c)} &= (n_c)\lambda_{c0} + (n - n_c - 1)\lambda_{p0}
\end{aligned}$$

which, along with Equations (DD.29)-(DD.33) implies that

$$\beta_{-c,p}^{SCP(n_c)} = \beta_{-p,p}^{SCP(n_c)} < \beta_{-p,c}^{SCP(n_c)} = \frac{1}{\sigma^2}\beta_{-p,0}^{SCP(n_c)}.$$

4. Combining Equations (DD.27) and (DD.28), we have

$$\begin{aligned}
\tilde{r}\left(\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)}\right) &= \tilde{r}\beta_p^{SCP(n_c)}\frac{n - 1 + \frac{n_c}{\sigma^2 + n_c}}{1 - \tilde{r}} \\
&= \tilde{r}\frac{n - 1 + \frac{n_c}{\sigma^2 + n_c}}{\sigma^2(n - \tilde{r}) + n(n_c + 1 - \tilde{r})} \\
&< \frac{n - 1 + \frac{n_c}{\sigma^2 + n_c}}{\sigma^2(n-1) + nn_c} = \frac{1}{\sigma^2 + n_c}
\end{aligned}$$

Therefore,

$$\lambda_{pc}^{SCP(n_c)} = \frac{1}{\sigma^2 + n_c + 1} + \tilde{r}\frac{1}{\sigma^2 + n_c + 1}\left(\beta_c^{SCP(n_c)} - \beta_p^{SCP(n_c)}\right)$$

$$
\begin{aligned}
&< \frac{1}{\sigma^2+n_c+1} + \frac{1}{(\sigma^2+n_c+1)(\sigma^2+n_c)} \\
&= \frac{1}{\sigma^2+n_c}
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_{-p,c}^{SCP(n_c)} &= \frac{1}{n-1}\left[n_c\lambda_{cc} + (n-n_c-1)\lambda_{pc}\right] \\
&= \frac{1}{n-1}\left[n_c\frac{1}{\sigma^2+n_c} + (n-n_c-1)\lambda_{pc}\right] \\
&< \frac{1}{n-1}\left[n_c\frac{1}{\sigma^2+n_c} + (n-n_c-1)\frac{1}{\sigma^2+n_c}\right] \\
&= \frac{1}{\sigma^2+n_c}
\end{aligned}
$$

5. Using the first item of this proof, we have that:

$$
\begin{aligned}
\beta_0^{SCP(n_c)} &= \sigma^2\frac{n-\tilde{r}+\frac{n_c}{\sigma^2+n_c}}{\sigma^2(n-\tilde{r})+n(n_c+1-\tilde{r})} \\
\beta_c^{SCP(n_c)} &= \frac{n-\tilde{r}+\frac{n_c}{\sigma^2+n_c}}{\sigma^2(n-\tilde{r})+n(n_c+1-\tilde{r})} \\
\beta_p^{SCP(n_c)} &= \frac{1-\tilde{r}}{\sigma^2(n-\tilde{r})+n(n_c+1-\tilde{r})}
\end{aligned}
$$

which imply:

$$
\begin{aligned}
\beta_0^{SCP(0)} &= \sigma^2\beta_p^{SCP(0)} = \sigma^2\frac{n-\tilde{r}}{\sigma^2(n-\tilde{r})+n(1-\tilde{r})} \\
\beta_c^{SCP(1)} - \beta_p^{SCP(1)} &= \frac{n-1+\frac{1}{\sigma^2+1}}{\sigma^2(n-\tilde{r})+n(2-\tilde{r})}
\end{aligned}
$$

From Equation (DD.10), we have that:

$$
\begin{aligned}
(n-1)\beta_{-p,0}^{SCP(1)} &= \lambda_{c0}^{SCP(1)} + (n-2)\lambda_{p0}^{SCP(1)} \\
(n-1)\beta_{-p,0}^{SCP(0)} &= (n-1)\lambda_{p0}^{SCP(0)} \\
(n-1)\beta_{-p,p}^{SCP(0)} &= \lambda_{pp}^{SCP(0)} \\
(n-1)\beta_{-p,c}^{SCP(1)} &= \lambda_{cc}^{SCP(1)} + (n-2)\lambda_{pc}^{SCP(1)}
\end{aligned}
$$

Using the previous item in this lemma, we can write λs as:

$$
\lambda_{c0}^{SCP(1)} = \frac{\sigma^2}{\sigma^2+1} \tag{DD.34}
$$

$$
\lambda_{cc}^{SCP(1)} = \frac{1}{\sigma^2+1} \tag{DD.35}
$$

$$
\lambda_{p0}^{SCP(1)} = \frac{\sigma^2}{\sigma^2+2} + \tilde{r}\frac{\sigma^2}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right) \tag{DD.36}
$$

$$\lambda_{p0}^{SCP(0)} = \frac{\sigma^2}{\sigma^2+1} + \tilde{r}\frac{1}{\sigma^2+1}\left(\beta_0^{SCP(0)} - \sigma^2\beta_p^{SCP(0)}\right) \tag{DD.37}$$

$$\lambda_{pc}^{SCP(1)} = \frac{1}{\sigma^2+2} + \tilde{r}\frac{1}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right) \tag{DD.38}$$

$$\lambda_{pp}^{SCP(0)} = \frac{1}{\sigma^2+1} - \tilde{r}\frac{1}{\sigma^2+1}\left(\beta_0^{SCP(0)} - \sigma^2\beta_p^{SCP(0)}\right) \tag{DD.39}$$

Using item 4 of this proof, notice that

$$\lambda_{p0}^{SCP(1)} < \frac{\sigma^2}{\sigma^2+2} + \frac{\sigma^2}{(\sigma^2+1)(\sigma^2+2)} = \frac{\sigma^2}{\sigma^2+1} = \lambda_{c0}^{SCP(1)}$$

Therefore,

$$\begin{aligned}
&\beta_{-p,0}^{SCP(0)} - \beta_{-p,0}^{SCP(1)} \\
&\quad = \frac{1}{n-1}\left[(n-1)\lambda_{p0}^{SCP(0)} - \left[\lambda_{c0}^{SCP(1)} + (n-2)\lambda_{p0}^{SCP(1)}\right]\right] \\
&\quad > \lambda_{p0}^{SCP(0)} - \lambda_{c0}^{SCP(1)} \\
&\quad = \frac{\sigma^2}{\sigma^2+1} + \tilde{r}\frac{1}{\sigma^2+1}\left(\beta_0^{SCP(0)} - \sigma^2\beta_p^{SCP(0)}\right) - \left[\frac{\sigma^2}{\sigma^2+2} + \tilde{r}\frac{\sigma^2}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right)\right] \\
&\quad = \frac{\sigma^2}{(\sigma^2+1)(\sigma^2+2)} + \tilde{r}\frac{1}{\sigma^2+1}\left(\beta_0^{SCP(0)} - \sigma^2\beta_p^{SCP(0)}\right) - \tilde{r}\frac{\sigma^2}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right) \\
&\quad = \frac{\sigma^2}{(\sigma^2+1)(\sigma^2+2)} - \tilde{r}\frac{\sigma^2}{\sigma^2+2}\frac{n-1+\frac{1}{\sigma^2+1}}{\sigma^2(n-\tilde{r})+n(2-\tilde{r})} \\
&\quad = \frac{\sigma^2}{(\sigma^2+1)(\sigma^2+2)}\left[1 - \tilde{r}\frac{(\sigma^2+1)(n-1)+1}{\sigma^2(n-\tilde{r})+n(2-\tilde{r})}\right] \\
&\quad = \frac{\sigma^2}{(\sigma^2+1)(\sigma^2+2)}\left[1 - \tilde{r}\frac{\sigma^2(n-1)+n}{\sigma^2(n-\tilde{r})+n(2-\tilde{r})}\right] \\
&\quad > 0
\end{aligned}$$

6. Using the expressions from the previous items:

$$\begin{aligned}
&\beta_{-p,c}^{SCP(1)} - \beta_{-p,p}^{SCP(0)} \\
&\quad = \frac{1}{n-1}\left[\lambda_{cc}^{SCP(1)} + (n-2)\lambda_{pc}^{SCP(1)} - \lambda_{pp}^{SCP(0)}\right] \\
&\quad = \frac{1}{n-1}\left[\frac{1}{\sigma^2+1} + (n-2)\left(\frac{1}{\sigma^2+2} + \tilde{r}\frac{1}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right)\right)\right. \\
&\qquad \left. - \left(\frac{1}{\sigma^2+1} - \tilde{r}\frac{1}{\sigma^2+1}\left(\beta_0^{SCP(0)} - \sigma^2\beta_p^{SCP(0)}\right)\right)\right] \\
&\quad = \frac{1}{n-1}\left[\frac{1}{\sigma^2+1} + (n-2)\left(\frac{1}{\sigma^2+2} + \tilde{r}\frac{1}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right)\right) - \frac{1}{\sigma^2+1}\right] \\
&\quad = \frac{n-2}{n-1}\left(\frac{1}{\sigma^2+2} + \tilde{r}\frac{1}{\sigma^2+2}\left(\beta_c^{SCP(1)} - \beta_p^{SCP(1)}\right)\right) \\
&\quad > 0
\end{aligned}$$

□

Lemma A.12. *For any core-periphery-observing-down network with $n_c - 1$ core players, $CPOD(n_c - 1)$, for every $n_c = 2, \ldots, n$, we have:*

1. $\beta_p^{CPOD(n_c-1)} < \beta_{n_c}^{CPOD(n_c-1)} < \beta_c^{CPOD(n_c-1)} = \frac{1}{\sigma^2}\beta_0^{CPOD(n_c-1)}$,
2. $\beta_{-c,p}^{CPOD(n_c-1)} = \beta_{-p,p}^{CPOD(n_c-1)} < \beta_{-c,n_c}^{CPOD(n_c-1)} < \beta_{-p,n_c}^{CPOD(n_c-1)}$,
3. $\beta_{-c,n_c}^{CPOD(n_c-1)} < \beta_{-p,c}^{CPOD(n_c-1)}$

Proof. This proof heavily rely on Equations (DD.5), (DD.7), and (DD.10) applied to the core-periphery-observing-down network with $n_c - 1$ core players.

1. From Equation (DD.7), we have that $CPOD(n_c - 1)$ betas satisfy:

$$\left[n - \tilde{r}\left(\overline{\mathcal{K}}_j + 1\right)\right]\beta_j = (1 - \tilde{r})\sum_{i=1}^{n}\tilde{g}_{ij} + \tilde{r}\sum_{i=1}^{n}\sum_{s=0}^{n}\beta_s(1 - g_{is})\tilde{g}_{ij}.$$

Hence, betas solve the following system of equations:

$$\begin{aligned}
[n - \tilde{r}n]\beta_0^{CPOD(n_c-1)} &= (1 - \tilde{r})\frac{\sigma^2}{\sigma^2 + n_c}n + \tilde{r}\frac{\sigma^2}{\sigma^2 + n_c}\left[n_c(n - n_c)\beta_p^{CPOD(n_c-1)}\right. && \text{(DD.40)}\\
&\quad \left. +(n - n_c)\left((n - n_c - 1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right)\right]\\
[n - \tilde{r}n]\beta_c^{CPOD(n_c-1)} &= (1 - \tilde{r})\frac{1}{\sigma^2 + n_c}n + \tilde{r}\frac{1}{\sigma^2 + n_c}\left[n_c(n - n_c)\beta_p^{CPOD(n_c-1)}\right. && \text{(DD.41)}\\
&\quad \left. +(n - n_c)\left((n - n_c - 1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right)\right]\\
[n - \tilde{r}n_c]\beta_{n_c}^{CPOD(n_c-1)} &= (1 - \tilde{r})\frac{1}{\sigma^2 + n_c}n_c + \tilde{r}\frac{1}{\sigma^2 + n_c}\left[n_c(n - n_c)\beta_p^{CPOD(n_c-1)}\right] && \text{(DD.42)}\\
[n - \tilde{r}]\beta_p^{CPOD(n_c-1)} &= (1 - \tilde{r})\frac{1}{\sigma^2 + n_c} && \text{(DD.43)}\\
&\quad + \tilde{r}\frac{(n - n_c - 1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}}{\sigma^2 + n_c}
\end{aligned}$$

By comparing Equations (DD.40) and (DD.41), we have that:

$$\beta_0^{CPOD(n_c-1)} = \sigma^2\beta_c^{CPOD(n_c-1)}.$$

We can use Equation (DD.42) to write Equation (DD.43) as follows:

$$\begin{aligned}
[n - \tilde{r}n_c]\beta_{n_c}^{CPOD(n_c-1)} &= (1 - \tilde{r})\frac{1}{\sigma^2 + n_c}n_c + \tilde{r}\frac{1}{\sigma^2 + n_c}n_c(n - n_c)\beta_p^{CPOD(n_c-1)}\\
&= \frac{n_c}{\sigma^2 + n_c}\left[1 - \tilde{r} + \tilde{r}\left[(n - n_c - 1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right]\right.\\
&\quad \left. +\tilde{r}\left(\beta_p^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)}\right)\right]\\
&= n_c\,[n - \tilde{r}]\beta_p^{CPOD(n_c-1)} - \frac{\tilde{r}}{\sigma^2 + n_c}\left(\beta_{n_c}^{CPOD(n_c-1)} - \beta_p^{CPOD(n_c-1)}\right)
\end{aligned}$$

solving for $\beta_{n_c}^{CPOD(n_c-1)}$:

$$\beta_{n_c}^{CPOD(n_c-1)} = \frac{n_c(n - \tilde{r}) + \frac{\tilde{r}}{\sigma^2+n_c}}{n - \tilde{r}n_c + \frac{\tilde{r}}{\sigma^2+n_c}}\beta_p^{CPOD(n_c-1)}$$

$$
\begin{aligned}
&= \underbrace{\frac{n_c(n-\tilde{r}) + \frac{\tilde{r}}{\sigma^2+n_c}}{n_c\left(\frac{n}{n_c}-\tilde{r}\right) + \frac{\tilde{r}}{\sigma^2+n_c}}}_{>1} \beta_p^{CPOD(n_c-1)} \\
&> \beta_p^{CPOD(n_c-1)}
\end{aligned}
$$

We can use Equations (DD.42) and (DD.43) to write Equation (DD.41) as follows:

$$
\begin{aligned}
[n-\tilde{r}n]\beta_c^{CPOD(n_c-1)} &= (1-\tilde{r})\frac{1}{\sigma^2+n_c}n + \tilde{r}\frac{1}{\sigma^2+n_c}\Big[n_c(n-n_c)\beta_p^{CPOD(n_c-1)} \\
&\quad +(n-n_c)\left((n-n_c-1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right)\Big] \\
&= (n-\tilde{r}n_c)\beta_{n_c}^{CPOD(n_c-1)} + (n-n_c)(n-\tilde{r})\beta_p^{CPOD(n_c-1)} \\
&= (n-\tilde{r}n)\beta_{n_c}^{CPOD(n_c-1)} + (n-n_c)\left[\tilde{r}\beta_{n_c}^{CPOD(n_c-1)} + (n-\tilde{r})\beta_p^{CPOD(n_c-1)}\right]
\end{aligned}
$$

solving for $\beta_c^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)}$:

$$
\beta_c^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)} = \frac{n-n_c}{n(1-\tilde{r})}\left[(n-\tilde{r})\beta_p^{CPOD(n_c-1)} + \tilde{r}\beta_{n_c}^{CPOD(n_c-1)}\right] > 0.
$$

2. From Equation (DD.10), we have that:

$$
\begin{aligned}
(n-1)\beta_{-c,n_c}^{CPOD(n_c-1)} &= (n_c-1)\lambda_{cn_c}^{CPOD(n_c-1)} \\
(n-1)\beta_{-c,p}^{CPOD(n_c-1)} &= \lambda_{pp}^{CPOD(n_c-1)} \\
(n-1)\beta_{-p,c}^{CPOD(n_c-1)} &= n_c\lambda_{cc}^{CPOD(n_c-1)} + (n-n_c-1)\lambda_{pc}^{CPOD(n_c-1)} \\
(n-1)\beta_{-p,n_c}^{CPOD(n_c-1)} &= n_c\lambda_{cn_c}^{CPOD(n_c-1)} \\
(n-1)\beta_{-p,p}^{CPOD(n_c-1)} &= \lambda_{pp}^{CPOD(n_c-1)}
\end{aligned}
$$

Hence, we already have that

$$
\beta_{-c,p}^{CPOD(n_c-1)} = \beta_{-p,p}^{CPOD(n_c-1)},
$$

and

$$
\beta_{-c,n_c}^{CPOD(n_c-1)} < \beta_{-p,n_c}^{CPOD(n_c-1)}.
$$

The λ's from Equation (DD.5) can be written as:

$$
\lambda_{ij} = \tilde{g}_{ij} + \tilde{r}\beta_j g_{ij} - \tilde{r}\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\tilde{g}_{ij}
$$

Under $CPOD(n_c-1)$, we have

$$
\begin{aligned}
\lambda_{cc}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\beta_c^{CPOD(n_c-1)} - \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_0^{CPOD(n_c-1)} + (n_c-1)\beta_c^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right) \\
\lambda_{cn_c}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\beta_{n_c}^{CPOD(n_c-1)} - \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_0^{CPOD(n_c-1)} + (n_c-1)\beta_c^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right) \\
\lambda_{pc}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\beta_c^{CPOD(n_c-1)} - \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_0^{CPOD(n_c-1)} + (n_c-1)\beta_c^{CPOD(n_c-1)} + \beta_p^{CPOD(n_c-1)}\right) \\
\lambda_{pp}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\beta_p^{CPOD(n_c-1)} - \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_0^{CPOD(n_c-1)} + (n_c-1)\beta_c^{CPOD(n_c-1)} + \beta_p^{CPOD(n_c-1)}\right)
\end{aligned}
$$

where $\lambda_{pp}^{CPOD(n_c-1)}$ is the weight of a peripheral agent own her own signal.

Notice that

$$
\begin{aligned}
&(n-1)\left[\beta_{-c,n_c}^{CPOD(n_c-1)} - \beta_{-p,p}^{CPOD(n_c-1)}\right] \\
&= (n_c-1)\lambda_{cn_c}^{CPOD(n_c-1)} - \lambda_{pp}^{CPOD(n_c-1)} \\
&> \lambda_{cn_c}^{CPOD(n_c-1)} - \lambda_{pp}^{CPOD(n_c-1)} \\
&= \tilde{r}\left[\beta_{n_c}^{CPOD(n_c-1)} - \beta_{p}^{CPOD(n_c-1)} - \frac{1}{\sigma^2+n_c}\left(\beta_{n_c}^{CPOD(n_c-1)} - \beta_{p}^{CPOD(n_c-1)}\right)\right] \\
&= \tilde{r}\frac{\sigma^2+n_c-1}{\sigma^2+n_c}\left(\beta_{n_c}^{CPOD(n_c-1)} - \beta_{p}^{CPOD(n_c-1)}\right) \\
&> 0,
\end{aligned}
$$

which is equivalent to

$$\beta_{-p,p}^{CPOD(n_c-1)} < \beta_{-c,n_c}^{CPOD(n_c-1)}.$$

3. The difference $\beta_{-p,c}^{CPOD(n_c-1)} - \beta_{-c,n_c}^{CPOD(n_c-1)}$ can be written as:

$$
\begin{aligned}
&(n-1)\left[\beta_{-p,c}^{CPOD(n_c-1)} - \beta_{-c,n_c}^{CPOD(n_c-1)}\right] \\
&= n_c\lambda_{cc}^{CPOD(n_c-1)} + (n-n_c-1)\lambda_{pc}^{CPOD(n_c-1)} - \left[(n_c-1)\lambda_{cn_c}^{CPOD(n_c-1)}\right] \\
&= \lambda_{cc}^{CPOD(n_c-1)} + (n-n_c-1)\lambda_{pc}^{CPOD(n_c-1)} + (n_c-1)\left[\lambda_{cc}^{CPOD(n_c-1)} - \lambda_{cn_c}^{CPOD(n_c-1)}\right] \\
&> (n_c-1)\left[\lambda_{cc}^{CPOD(n_c-1)} - \lambda_{cn_c}^{CPOD(n_c-1)}\right] \\
&= (n_c-1)\tilde{r}\left[\beta_{c}^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)}\right] \\
&> 0.
\end{aligned}
$$

which is equivalent to

$$\beta_{-c,n_c}^{CPOD(n_c-1)} < \beta_{-p,c}^{CPOD(n_c-1)}.$$

□

Lemma A.13. *For a simple-core-periphery network with n_c core players, $SCP(n_c)$, a core-periphery-observing-down network with n_c-1 core players, for every $n_c = 2,\dots,n-1$, we have:*

1. $\beta_{p}^{SCP(n_c)} < \beta_{p}^{CPOD(n_c-1)}$,
2. $\beta_{c}^{SCP(n_c)} < \beta_{c}^{CPOD(n_c-1)}$,
3. $\beta_{-p,c}^{SCP(n_c)} < \beta_{-p,c}^{CPOD(n_c-1)}$,
4. $\beta_{-p,0}^{SCP(n_c)} < \beta_{-p,0}^{CPOD(n_c-1)}$,
5. $\beta_{-p,n_c}^{CPOD(n_c-1)} < \beta_{-p,c}^{SCP(n_c)}$,
6. $\beta_{p}^{SPC(n_c-1)} < \beta_{p}^{CPOD(n_c-1)}$,
7. $\beta_{c}^{CPOD(n_c-1)} < \beta_{c}^{SPC(n_c-1)}$,
8. $\beta_{-c,c}^{CPOD(n_c-1)} < \beta_{-c,c}^{SCP(n_c-1)}$,
9. $\beta_{-c,0}^{CPOD(n_c-1)} < \beta_{-c,0}^{SCP(n_c-1)}$,

10. $\beta_{-c,p}^{SCP(n_c-1)} < \beta_{-c,n_c}^{CPOD(n_c-1)}$,

Proof. This proof uses derivations available in the proofs of Lemmata A.11 and A.12.

1. From Equation DD.43,

$$
\begin{aligned}
[n-\tilde{r}]\beta_p^{CPOD(n_c-1)} &= (1-\tilde{r})\frac{1}{\sigma^2+n_c} + \tilde{r}\frac{(n-n_c-1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}}{\sigma^2+n_c} \\
&> (1-\tilde{r})\frac{1}{\sigma^2+n_c} + \tilde{r}\frac{(n-n_c-1)\beta_p^{CPOD(n_c-1)}}{\sigma^2+n_c} \\
&> (1-\tilde{r})\frac{1}{\sigma^2+n_c+1} + \tilde{r}\frac{(n-n_c-1)\beta_p^{CPOD(n_c-1)}}{\sigma^2+n_c+1}
\end{aligned}
$$

which implies that

$$
\beta_p^{CPOD(n_c-1)} > \frac{1-\tilde{r}}{\sigma^2(n-\tilde{r}) + n(n_c+1-\tilde{r})} = \beta_p^{SPC(n_c)},
$$

where the the last equality is from Equation (DD.27).

2. From Equation (DD.41)

$$
\begin{aligned}
[n-\tilde{r}n]\beta_c^{CPOD(n_c-1)} &= (1-\tilde{r})\frac{1}{\sigma^2+n_c}n + \tilde{r}\frac{1}{\sigma^2+n_c}\Big[n_c(n-n_c)\beta_p^{CPOD(n_c-1)} \\
&\quad +(n-n_c)\left((n-n_c-1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}\right)\Big] \\
&> (1-\tilde{r})\frac{1}{\sigma^2+n_c}n + \tilde{r}\frac{1}{\sigma^2+n_c}\Big[n_c(n-n_c)\beta_p^{CPOD(n_c-1)} \\
&\quad +(n-n_c)(n-n_c-1)\beta_p^{CPOD(n_c-1)}\Big] \\
&= (1-\tilde{r})\left[\frac{n_c}{\sigma^2+n_c} + \frac{n-n_c}{\sigma^2+n_c}\right] + \tilde{r}\Big[n_c\frac{n-n_c}{\sigma^2+n_c}\beta_p^{CPOD(n_c-1)} \\
&\quad +\frac{n-n_c}{\sigma^2+n_c}\left((n-n_c-1)\beta_p^{CPOD(n_c-1)}\right)\Big] \\
&> (1-\tilde{r})\left[\frac{n_c}{\sigma^2+n_c} + \frac{n-n_c}{\sigma^2+n_c+1}\right] + \tilde{r}\Big[n_c\frac{n-n_c}{\sigma^2+n_c}\beta_p^{CPOD(n_c-1)} \\
&\quad +\frac{n-n_c}{\sigma^2+n_c+1}\left((n-n_c-1)\beta_p^{CPOD(n_c-1)}\right)\Big]
\end{aligned}
$$

using the previous item

$$
\begin{aligned}
&> (1-\tilde{r})\left[\frac{n_c}{\sigma^2+n_c} + \frac{n-n_c}{\sigma^2+n_c+1}\right] + \tilde{r}\Big[n_c\frac{n-n_c}{\sigma^2+n_c}\beta_p^{SPC(n_c)} \\
&\quad +\frac{n-n_c}{\sigma^2+n_c+1}\left((n-n_c-1)\beta_p^{SPC(n_c)}\right)\Big]
\end{aligned}
$$

using Equation (DD.25)

$$
= [n-\tilde{r}n]\beta_c^{SPC(n_c)}.
$$

Therefore, $\beta_c^{CPOD(n_c-1)} > \beta_c^{SPC(n_c)}$.

3. From Equation (DD.10), we can write

$$(n-1)\beta_{-p,c}^{SCP(n_c)} = n_c\lambda_{cc}^{SCP(n_c)} + (n-n_c-1)\lambda_{pc}^{SCP(n_c)}$$
$$(n-1)\beta_{-p,c}^{CPOD(n_c-1)} = n_c\lambda_{cc}^{CPOD(n_c-1)} + (n-n_c-1)\lambda_{pc}^{CPOD(n_c-1)}$$

Therefore, it is sufficient to show that $\lambda_{cc}^{SCP(n_c)} < \lambda_{cc}^{CPOD(n_c-1)}$ and $\lambda_{pc}^{SCP(n_c)} < \lambda_{pc}^{CPOD(n_c-1)}$. Using Equation (DD.5) and following derivations similar to previous lemma, notice that

$$\begin{aligned}
\lambda_{cc}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)}\right)\\
\lambda_{pc}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_p^{CPOD(n_c-1)}\right)\\
\lambda_{cc}^{SCP(n_c)} &= \frac{1}{\sigma^2+n_c}\\
\lambda_{pc}^{SCP(n_c)} &= \frac{1}{\sigma^2+n_c+1} + \tilde{r}\frac{1}{\sigma^2+n_c+1}\left(\beta_c^{SPC(n_c)} - \beta_p^{SPC(n_c)}\right)
\end{aligned}$$

From Lemma A.12, $\beta_c^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)} > 0$, thus $\lambda_{cc}^{SCP(n_c)} < \lambda_{cc}^{CPOD(n_c-1)}$.
To show that $\lambda_{pc}^{SCP(n_c)} < \lambda_{pc}^{CPOD(n_c-1)}$, notice that we can write

$$\begin{aligned}
\lambda_{pc}^{SCP(n_c)} &= \frac{1-\tilde{r}}{\sigma^2+n_c+1} + \tilde{r}\frac{1}{\sigma^2+n_c+1}\left((\sigma^2+n_c+1)\beta_c^{SPC(n_c)} + (n-n_c-1)\beta_p^{SPC(n_c)}\right)\\
&= \frac{1-\tilde{r}}{\sigma^2+n_c+1} + \tilde{r}\left(\beta_c^{SPC(n_c)} + \frac{n-n_c-1}{\sigma^2+n_c+1}\beta_p^{SPC(n_c)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{pc}^{CPOD(n_c-1)} &= \frac{1-\tilde{r}}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left((n_c+\sigma^2)\beta_c^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)} + (n-n_c-1)\beta_p^{CPOD(n_c-1)}\right)\\
&= \frac{1-\tilde{r}}{\sigma^2+n_c} + \tilde{r}\left(\beta_c^{CPOD(n_c-1)} + \frac{\beta_{n_c}^{CPOD(n_c-1)}}{\sigma^2+n_c} + \frac{n-n_c-1}{\sigma^2+n_c}\beta_p^{CPOD(n_c-1)}\right)\\
&> \frac{1-\tilde{r}}{\sigma^2+n_c} + \tilde{r}\left(\beta_c^{CPOD(n_c-1)} + \frac{n-n_c-1}{\sigma^2+n_c}\beta_p^{CPOD(n_c-1)}\right)\\
&> \frac{1-\tilde{r}}{\sigma^2+n_c+1} + \tilde{r}\left(\beta_c^{CPOD(n_c-1)} + \frac{n-n_c-1}{\sigma^2+n_c+1}\beta_p^{CPOD(n_c-1)}\right)
\end{aligned}$$

Hence to show that $\lambda_{pc}^{SCP(n_c)} < \lambda_{pc}^{CPOD(n_c-1)}$, it is sufficient to that $\beta_c^{SPC(n_c)} < \beta_c^{CPOD(n_c-1)}$ and $\beta_p^{SPC(n_c)} < \beta_p^{CPOD(n_c-1)}$, which is demonstrated in the previous item.

4. Using Lemmata A.11 and A.12, along with the previous item, we have that

$$\beta_{-p,0}^{SCP(n_c)} = \sigma^2\beta_{-p,c}^{SCP(n_c)} < \sigma^2\beta_{-p,c}^{CPOD(n_c)} = \beta_{-p,0}^{CPOD(n_c-1)}.$$

5. Since betas sum to one from Equation (DD.12), that is,

$$1 = \beta_{-p,n_c}^{CPOD(n_c-1)} + \beta_{-p,0}^{CPOD(n_c-1)} + (n_c-1)\beta_{-p,c}^{CPOD(n_c-1)} + (n-n_c)\beta_{-p,p}^{CPOD(n_c-1)}$$

we have using the previous items that

$$\begin{aligned}
\beta_{-p,n_c}^{CPOD(n_c-1)} &= 1 - \beta_{-p,0}^{CPOD(n_c-1)} - (n_c - 1)\beta_{-p,c}^{CPOD(n_c-1)} - (n - n_c)\beta_{-p,p}^{CPOD(n_c-1)} \\
&= 1 - (\sigma^2 + n_c - 1)\beta_{-p,c}^{CPOD(n_c-1)} - (n - n_c)\beta_{-p,p}^{CPOD(n_c-1)} \\
&< 1 - (\sigma^2 + n_c - 1)\beta_{-p,c}^{SPC(n_c)} - (n - n_c)\beta_{-p,p}^{SPC(n_c)} \\
&= 1 - (\sigma^2 + n_c)\beta_{-p,c}^{SPC(n_c)} - (n - n_c)\beta_{-p,p}^{SPC(n_c)} + \beta_{-p,c}^{SPC(n_c)} \\
&= \beta_{-p,c}^{SPC(n_c)}
\end{aligned}$$

6. From Equation DD.43,

$$\begin{aligned}
[n - \tilde{r}]\beta_p^{CPOD(n_c-1)} &= (1 - \tilde{r})\frac{1}{\sigma^2 + n_c} + \tilde{r}\frac{(n - n_c - 1)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)}}{\sigma^2 + n_c} \\
&= (1 - \tilde{r})\frac{1}{\sigma^2 + n_c} + \tilde{r}\frac{(n - n_c)\beta_p^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)} - \beta_p^{CPOD(n_c-1)}}{\sigma^2 + n_c} \\
&> (1 - \tilde{r})\frac{1}{\sigma^2 + n_c} + \tilde{r}\frac{(n - n_c)\beta_p^{CPOD(n_c-1)}}{\sigma^2 + n_c}
\end{aligned}$$

which implies that

$$\beta_p^{CPOD(n_c-1)} > \frac{1 - \tilde{r}}{\sigma^2(n - \tilde{r}) + n(n_c - \tilde{r})} = \beta_p^{SPC(n_c-1)},$$

where the the last equality is from Equation (DD.27).

7. Using the fact the betas sum to one from Equation (DD.3), that is,

$$1 = (\sigma^2 + n_c - 1)\beta_c^{CPOD(n_c-1)} + \beta_{n_c}^{CPOD(n_c-1)} + (n - n_c)\beta_p^{CPOD(n_c-1)}$$

we have:

$$(\sigma^2 + n_c - 1)\beta_c^{CPOD(n_c-1)} = 1 - \beta_{n_c}^{CPOD(n_c-1)} - (n - n_c)\beta_p^{CPOD(n_c-1)}$$

using Lemma A.12

$$< 1 - (n - n_c - 1)\beta_p^{CPOD(n_c-1)}$$

the previous item

$$\begin{aligned}
&< 1 - (n - n_c - 1)\beta_p^{SPC(n_c-1)} \\
&= (\sigma^2 + n_c - 1)\beta_c^{SPC(n_c-1)}.
\end{aligned}$$

Therefore $\beta_c^{CPOD(n_c-1)} < \beta_c^{SPC(n_c-1)}$.

8. From Equation (DD.10), we can write

$$
\begin{aligned}
(n-1)\beta_{-c,c}^{CPOD(n_c-1)} &= (n_c-1)\lambda_{cc}^{CPOD(n_c-1)} + (n-n_c)\lambda_{pc}^{CPOD(n_c-1)} \\
(n-1)\beta_{-c,c}^{SCP(n_c-1)} &= (n_c-2)\lambda_{cc}^{SCP(n_c-1)} + (n-n_c+1)\lambda_{pc}^{SCP(n_c-1)}
\end{aligned}
$$

Using Equation (DD.5) and following derivations similar to previous lemma, we have

$$
\begin{aligned}
\lambda_{cc}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)}\right) \\
\lambda_{pc}^{CPOD(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_{p}^{CPOD(n_c-1)}\right) \\
\lambda_{cc}^{SCP(n_c-1)} &= \frac{1}{\sigma^2+n_c-1} \\
\lambda_{pc}^{SCP(n_c-1)} &= \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{SPC(n_c-1)} - \beta_{p}^{SPC(n_c-1)}\right)
\end{aligned}
$$

We can show that $\lambda_{pc}^{SCP(n_c)} < \lambda_{cc}^{SCP(n_c)}$:

$$
\lambda_{pc}^{SCP(n_c-1)} = \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{SPC(n_c-1)} - \beta_{p}^{SPC(n_c-1)}\right)
$$

using Lemma A.11

$$
\begin{aligned}
&< \frac{1}{\sigma^2+n_c} + \frac{1}{(\sigma^2+n_c)(\sigma^2+n_c-1)} \\
&= \frac{1}{\sigma^2+n_c-1} \\
&= \lambda_{cc}^{SCP(n_c-1)}.
\end{aligned}
$$

We can also show that $\lambda_{cc}^{CPOD(n_c-1)} < \lambda_{pc}^{CPOD(n_c-1)}$:

$$
\lambda_{cc}^{CPOD(n_c-1)} = \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_{n_c}^{CPOD(n_c-1)}\right)
$$

using Lemma A.12

$$
\begin{aligned}
&< \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_{p}^{SPC(n_c-1)}\right) \\
&= \lambda_{pc}^{CPOD(n_c-1)}
\end{aligned}
$$

and that $\lambda_{pc}^{CPOD(n_c-1)} < \lambda_{pc}^{SCP(n_c-1)}$

$$
\lambda_{pc}^{CPOD(n_c-1)} = \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{CPOD(n_c-1)} - \beta_{p}^{SPC(n_c-1)}\right)
$$

using the previous items

$$
< \frac{1}{\sigma^2+n_c} + \tilde{r}\frac{1}{\sigma^2+n_c}\left(\beta_c^{SPC(n_c-1)} - \beta_{p}^{SPC(n_c-1)}\right)
$$

$$= \lambda_{pc}^{SCP(n_c-1)}.$$

Hence, we have that

$$\begin{aligned}
(n-1)\beta_{-c,c}^{SCP(n_c-1)} &= (n_c-2)\lambda_{cc}^{SCP(n_c-1)} + (n-n_c+1)\lambda_{pc}^{SCP(n_c-1)} \\
&> (n_c-2)\lambda_{pc}^{SCP(n_c-1)} + (n-n_c+1)\lambda_{pc}^{SCP(n_c-1)} \\
&= (n-1)\lambda_{pc}^{SCP(n_c-1)} \\
&> (n-1)\lambda_{pc}^{CPOD(n_c-1)} \\
&> (n_c-1)\lambda_{cc}^{CPOD(n_c-1)} + (n-n_c)\lambda_{pc}^{CPOD(n_c-1)} \\
&= (n-1)\beta_{-c,c}^{CPOD(n_c-1)}.
\end{aligned}$$

9. Using Lemmata A.11 and A.12, along with the previous item:

$$\beta_{-c,0}^{CPOD(n_c-1)} = \sigma^2\beta_{-c,c}^{CPOD(n_c-1)} < \sigma^2\beta_{-c,c}^{SCP(n_c-1)} = \beta_{-c,0}^{SCP(n_c-1)}.$$

10. From Equation (DD.10), we can write

$$\begin{aligned}
(n-1)\beta_{-c,p}^{SCP(n_c-1)} &= \lambda_{pp}^{SCP(n_c-1)} \\
(n-1)\beta_{-c,n_c}^{CPOD(n_c-1)} &= (n_c-1)\lambda_{cn_c}^{CPOD(n_c-1)}
\end{aligned}$$

Using Equation (DD.5) and following derivations similar to previous lemma, we have

$$\lambda_{cn_c}^{CPOD(n_c-1)} = \frac{1}{\sigma^2+n_c} + \tilde{r}\beta_{n_c}^{CPOD(n_c-1)} - \tilde{r}\frac{1}{\sigma^2+n_c}\left(1-(n-n_c)\beta_p^{CPOD(n_c-1)}\right)$$

using Lemma A.12

$$> \frac{1-\tilde{r}}{\sigma^2+n_c} + \tilde{r}\beta_p^{CPOD(n_c-1)} + \tilde{r}\frac{n-n_c}{\sigma^2+n_c}\beta_p^{CPOD(n_c-1)}$$

using previous item

$$> \frac{1-\tilde{r}}{\sigma^2+n_c} + \tilde{r}\beta_p^{SPC(n_c-1)} + \tilde{r}\frac{n-n_c}{\sigma^2+n_c}\beta_p^{SPC(n_c-1)}$$

using Equations (DD.5) and (DD.33)

$$= \lambda_{pp}^{SPC(n_c-1)}$$

Therefore,

$$(n-1)\beta_{-c,n_c}^{CPOD(n_c-1)} = (n_c-1)\lambda_{cn_c}^{CPOD(n_c-1)} > \lambda_{cn_c}^{CPOD(n_c-1)} > \lambda_{pp}^{SCP(n_c-1)} = (n-1)\beta_{-c,p}^{SCP(n_c-1)}.$$

□

C.2 Possible Deviations

The first step of our proof is to understand which possible deviations may bind behavior at each possible equilibrium candidate. We know that agents choose their connections in order to maximize their expected payoff.

Before we focus on core-periphery networks, let us show how if the marginal cost of acquiring signals is weakly increasing, Assumption 2, the payoff function obtained in Theorem 2 limits the relevant deviations that need to be checked in order to test whether a network is an equilibrium or not.

Lemma A.14. *Given Assumption 2, for any sequence $\{\beta_{-i,j}\}_{j=1}^{n}$, the marginal benefit of observing an additional signal weakly decreases with the number of signals observed.*

Proof. Given Lemma A.1, we know that in any best response to any sequence $\{\beta_{-i,j}\}_{j=1}^{n}$, if agent i observes m signals, he observes the m signals with the highest $\beta_{-i,j}$, not including his own signal. Agents rank the signals by their influence and observe them from top to bottom. For simplicity of notation let us order the agents by their $\beta_{-i,j}$, $\beta_{-i,1} \geq \beta_{-i,2} \geq ... \geq \beta_{-i,n}$.

Let us recall the expression obtained in Theorem 2. Given that we know the order of connections an agent must follow, to show the lemma above, it suffices to show that the payoff function has weakly decreasing marginal benefits of connections.

$$-\frac{\sigma^2}{\sigma^2 + \mathcal{K}_i + 1}\left(1 - r\sum_{j=0}^{n} g_{ij}\beta_{-i,j}\right)^2 - r^2\sigma^2\sum_{j=0}^{n}(1 - g_{ij})\beta_{-i,j}^2 - c\left(\mathcal{K}_i\right)$$

The payoff expression is a sum of three different functions. The last two trivially have weakly decreasing marginal benefits, as $c(\mathcal{K}_i)$ has increasing marginal costs, and $-r^2\sigma^2\sum_{j=0}^{n}(1 - g_{ij})\beta_{-i,j}^2 = -r^2\sigma^2 Z + \sum_{j=0}^{n} g_{ij}\beta_{-i,j}^2$, where Z is a constant and $\beta_{-i,j}$ is weakly decreasing with the number of signals observed.

Let us consider the first term of the function, $T_1 = -\frac{\sigma^2}{\sigma^2+\mathcal{K}_i+1}\left(1 - r\sum_{j=0} g_{ij}\beta_{-i,j}\right)^2$. Given a set of signals agent i is observing (holding $g_{i,j}$ fixed), let us consider the deviation of observing the set D, with additional d signals. Note that d can be positive or negative, representing forming or breaking links. The first term of the payoff of doing so is given by $T_1(d) = -\frac{\sigma^2}{\sigma^2+\mathcal{K}_i+1+d}\left(1 - r\sum_{j=0} g_{ij}\beta_{-i,j} - rdB_d\right)^2$, where B_d is the average $\beta_{-i,j}$ in the set D. The first derivative of $T_1(d)$ with respect to d is given by:

$$\frac{\partial T_1(d)}{\partial d} = \frac{1 - r\sum_{j=0} g_{ij}\beta_{-i,j} - rdB_d}{\sigma^2 + \mathcal{K}_i + 1 + d}2\sigma^2\frac{\partial dB_d}{\partial d} + \left(\frac{1 - r\sum_{j=0} g_{ij}\beta_{-i,j} - rdB_d}{\sigma^2 + \mathcal{K}_i + 1 + d}\right)^2\sigma^2$$

Note that the second derivative is obviously negative, as d raises the denominators of both strictly positive terms, and as dB_d is the sum of influences—which is positive, increasing, and concave by definition. □

The central implication of Lemma A.14 is that, given Assumption 2, to check whether any network is an equilibrium it suffices to check at most 2 deviations per agent: (i) to stop observing the least influential signal (smaller $\beta_{-i,j}$) the agent is observing and (ii) to observe the most influential signal (higher $\beta_{-i,j}$) the agent is not observing.

Let us now move to consider core-periphery networks and to further restrict the set of deviations that need to be checked when evaluating whether a certain network is an equilibrium.

SCP networks First, consider a simple-core-periphery network with n_c core players, $SCP(n_c)$. An agent is either in the core or in the periphery of the network. All core agents have the same expected payoff, furthermore all core agents would receive the same expected payoff if they did a particular deviation to either break or form one link. Also, all agents in the periphery have the same expected payoff, and all of them would receive the same expected payoff if they did a particular deviation. That is, if a core player in a $SCP(n_c)$ network would observe one extra signal from a periphery agent, her payoff would change from $\Pi_c(n_c) = U_c(n_c) - C(n_c - 1)$ to $U_{cf}(n_c) - C(n_c)$, where $U_{cf}(n_c)$ represents the core-player utility from deviating from a simple-core-periphery network with n_c players in the core and forming a link. That payoff change is independent of which core agent deviates and of which extra signal he decides to observe.

Let us formally define all possible deviations.[8]The core may deviate to break a link or to form a link. And the periphery may deviate to break or to form a link:

$$\Delta\Pi^{SCP}(n_c) = \begin{cases} \Delta\Pi^{SCP}_{cb}(n_c) & \text{core agent breaks a link} \\ \Delta\Pi^{SCP}_{cf}(n_c) & \text{core agent forms a link} \\ \Delta\Pi^{SCP}_{pb}(n_c) & \text{periphery agent breaks a link} \\ \Delta\Pi^{SCP}_{pf}(n_c) & \text{periphery agent forms a link} \end{cases}$$

$$\begin{aligned} \Delta\Pi^{SCP}_{cb}(n_c) &= (U_{cb}(n_c) - C(n_c - 2)) - (U_c(n_c) - C(n_c - 1)) = \Delta U^{SCP}_{cb}(n_c) + c_{n_c-1} \\ \Delta\Pi^{SCP}_{cf}(n_c) &= (U_{cf}(n_c) - C(n_c)) - (U_c(n_c) - C(n_c - 1)) = \Delta U^{SCP}_{cf}(n_c) - c_{n_c} \\ \Delta\Pi^{SCP}_{pb}(n_c) &= (U_{pb}(n_c) - C(n_c - 1)) - (U_p(n_c) - C(n_c)) = \Delta U^{SCP}_{pb}(n_c) + c_{n_c} \\ \Delta\Pi^{SCP}_{pf}(n_c) &= (U_{pf}(n_c) - C(n_c + 1)) - (U_p(n_c) - C(n_c)) = \Delta U^{SCP}_{pf}(n_c) - c_{n_c+1} \end{aligned}$$

Using concavity of the payoff structure and the fact that the periphery agent has more information than the core agent, we have that intuitively the gain from acquiring another signal must be larger for the core agent than for the periphery one. Let us define the the sets of deviations follows:

$$\begin{aligned} D^{SCP(n_c)}_{pf} &= \text{additional signal observed by peripheral agent when forming new link} \\ D^{SCP(n_c)}_{cf} &= \text{additional signal observed by core agent when forming new link} \\ D^{SCP(n_c)}_{pb} &= \text{signal not observed by peripheral agent when breaking an existing link} \\ D^{SCP(n_c)}_{cb} &= \text{signal not observed by core agent when breaking an existing link} \end{aligned}$$

Given the structure of the SCP network, $D^{SCP(n_c)}_{pf}$ and $D^{SCP(n_c)}_{cf}$ consist of a signal from the periphery, while $D^{SCP(n_c)}_{pb}$ and $D^{SCP(n_c)}_{cb}$ consist of a signal from the core. Formally, Lemma A.15 shows that $\Delta U^{SCP}_{cf}(n_c) \geq \Delta U^{SCP}_{pf}(n_c)$.

Lemma A.15. *For any simple-core-periphery network with n_c core players, $SCP(n_c)$, we have $\Delta U^{SCP}_{pf}(n_c) < \Delta U^{SCP}_{cf}(n_c)$ for every $n_c = 1, \dots, n-1$.*

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U^{SPC}_{pf}(n_c) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D^{SCP(n_c)}_{pf}} g^{SCP(n_c)}_{p,j}\beta^{SCP(n_c)}_{-p,j} - r\sum_{j\in D^{SCP(n_c)}_{pf}} \beta^{SCP(n_c)}_{-p,j}\right)^2}{\sigma^2 + n_c + 2}}_{\text{Observing } D^{SCP(n_c)}_{pf}}$$

[8]The possible deviations are also bound by the number of agents, for instance if $n_c = n$ it is not possible for the periphery agent to form any other link. Or if $n_c = 1$ it is not possible for the core agent to stop observing a signal, as she is not observing any signal.

$$
\begin{aligned}
&+\underbrace{\frac{\sigma^2\left(1-r\sum_{j=0}^{n} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)^2}{\sigma^2+n_c+1}}_{\text{Without observing}}\\
&-\underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{p,j}^{SCP(n_c)})\beta_{-p,j}^{SCP(n_c)2}-\sum_{j\in D_{pf}^{SCP(n_c)}}\beta_{-p,j}^{SCP(n_c)2}\right)}_{\text{Observing } D_{pf}^{SCP(n_c)}}\\
&+\underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{p,j}^{SCP(n_c)})\beta_{-p,j}^{SCP(n_c)2}\right)}_{\text{Without observing}}\\
=&\left(1-r\sum_{j\notin D_{pf}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c+1}-\frac{\sigma^2}{\sigma^2+n_c+2}\right)\\
&+2\frac{r\sigma^2}{\sigma^2+n_c+2}\left(1-r\sum_{j\notin D_{pf}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)\beta_{-p,p}^{SCP(n_c)}\\
&+r^2\sigma^2\beta_{-p,p}^{SCP(n_c)2}\left[1-\frac{1}{\sigma^2+n_c+2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta U_{cf}^{SCP}(n_c-1)=&-\underbrace{\frac{\sigma^2\left(1-r\sum_{j\notin D_{cf}^{SCP(n_c)}} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)}-r\sum_{j\in D_{cf}^{SCP(n_c)}}\beta_{-c,j}^{SCP(n_c)}\right)^2}{\sigma^2+n_c+1}}_{\text{Observing } D_{cf}^{SCP(n_c)}}\\
&+\underbrace{\frac{\sigma^2\left(1-r\sum_{j=0}^{n} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)}\right)^2}{\sigma^2+n_c}}_{\text{Without observing}}\\
&-\underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{SCP(n_c)})\beta_{-c,j}^{SCP(n_c)2}-\sum_{j\in D_{cf}^{SCP(n_c)}}\beta_{-c,j}^{SCP(n_c)2}\right)}_{\text{Observing } D_{cf}^{SCP(n_c)}}\\
&+\underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{SCP(n_c)})\beta_{-c,j}^{SCP(n_c)2}\right)}_{\text{Without observing}}\\
=&\left(1-r\sum_{j\notin D_{cf}^{SCP(n_c)}} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c}-\frac{\sigma^2}{\sigma^2+n_c+1}\right)\\
&+2\frac{r\sigma^2}{\sigma^2+n_c+1}\left(1-r\sum_{j\notin D_{cf}^{SCP(n_c)}} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)}\right)\beta_{-c,p}^{SCP(n_c)}\\
&+r^2\sigma^2\beta_{-c,p}^{SCP(n_c)2}\left[1-\frac{1}{\sigma^2+n_c+1}\right]
\end{aligned}
$$

Hence, to prove that $\Delta U^{SCP}_{pf}(n_c) < \Delta U^{SCP}_{cf}(n_c)$ it is sufficient to show that:

(i) $\sum_{j \notin D^{SCP(n_c)}_{cf}} g^{SCP(n_c)}_{c,j} \beta^{SCP(n_c)}_{-c,j} < \sum_{j \notin D^{SPC(n_c)}_{pf}} g^{SCP(n_c)}_{p,j} \beta^{SCP(n_c)}_{-p,j}$, and

(ii) $\beta^{SCP(n_c)}_{-c,p} \leq \beta^{SCP(n_c)}_{-p,p}$.

Notice that:

$$\begin{aligned} \sum_{j \notin D^{SCP(n_c)}_{pf}} g^{SCP(n_c)}_{p,j} \beta^{SCP(n_c)}_{-p,j} &= \beta^{SCP(n_c)}_{-p,0} + n_c \beta^{SCP(n_c)}_{-p,c} \\ &= 1 - (n - n_c - 1) \beta^{SCP(n_c)}_{-p,p} \end{aligned}$$

and

$$\begin{aligned} \sum_{j \notin D^{SCP(n_c)}_{cf}} g^{SCP(n_c)}_{c,j} \beta^{SCP(n_c)}_{-c,j} &= \beta^{SCP(n_c)}_{-c,0} + n_c \beta^{SCP(n_c)}_{-c,c} \\ &= 1 - (n - n_c) \beta^{SCP(n_c-1)}_{-c,p} \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{j \notin D^{SCP(n_c)}_{pf}} g^{SCP(n_c)}_{p,j} \beta^{SCP(n_c)}_{-p,j} - \sum_{j \notin D^{SCP(n_c)}_{cf}} g^{SCP(n_c)}_{c,j} \beta^{SCP(n_c)}_{-c,j} \\ &= (n - n_c) \underbrace{\left[\beta^{SCP(n_c)}_{-c,p} - \beta^{SCP(n_c)}_{-p,p}\right]}_{=0 \text{ (Lemma A.11)}} + \underbrace{\beta^{SCP(n_c)}_{-p,p}}_{>0} > 0. \end{aligned}$$

Furthermore, from Lemma A.11, we know that $\beta^{SCP(n_c)}_{-c,p} = \beta^{SCP(n_c)}_{-p,p}$, which guarantees the second sufficient conditions mentioned above. □

At the same time, again by the concavity of the payoff structure and the periphery agent information set being a strict superset of the core agent's information set, the loss from not acquiring a signal is less severe for the periphery agent. Formally, Lemma A.16 shows that $\Delta U^{SCP}_{pb}(n_c) \geq \Delta U^{SCP}_{cb}(n_c)$.

Lemma A.16. *For any simple-core-periphery network with n_c core players, $SCP(n_c)$, we have $\Delta U^{SCP}_{cb}(n_c) < \Delta U^{SCP}_{pb}(n_c)$ for every $n_c = 1, \ldots, n-1$.*

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\begin{aligned} \Delta U^{SCP}_{pb}(n_c) = &- \underbrace{\frac{\sigma^2 \left(1 - r \sum_{j \notin D^{SCP(n_c)}_{pb}} g^{SCP(n_c)}_{p,j} \beta^{SCP(n_c)}_{-p,j}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2 \left(1 - r \sum_{j=0}^{n} g^{SCP(n_c)}_{p,j} \beta^{SCP(n_c)}_{-p,j}\right)^2}{\sigma^2 + n_c + 1}}_{\text{Observing} D^{SCP(n_c)}_{pb}} \\ &\underbrace{- r^2 \sigma^2 \left(\sum_{j=0}^{n} (1 - g^{SCP(n_c)}_{p,j}) \beta^{SCP(n_c)}_{-p,j}{}^2 + \sum_{j \in D^{SCP(n_c)}_{pb}} \beta^{SCP(n_c)}_{-p,j}{}^2 \right)}_{\text{Without observing}} \end{aligned}$$

$$\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{p,j}^{SCP(n_c)})\beta_{-p,j}^{SCP(n_c)2}}_{\text{Observing}D_{pb}^{SCP(n_c)}},$$

and

$$\Delta U_{cb}^{SCP}(n_c) = -\underbrace{\frac{\sigma^2\left(1-r\sum_{j\notin D_{cb}^{SCP(n_c)}} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)}\right)^2}{\sigma^2+n_c-1}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2\left(1-r\sum_{j=0}^{n} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)}\right)^2}{\sigma^2+n_c}}_{\text{Observing}D_{cb}^{SCP(n_c)}}$$
$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{SCP(n_c)})\beta_{-c,j}^{SCP(n_c)2}+\sum_{j\in D_{cb}^{SCP(n_c)}}\beta_{-c,j}^{SCP(n_c)2}\right)}_{\text{Without observing}}$$
$$\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{c,j}^{SCP(n_c)})\beta_{-c,j}^{SCP(n_c)2}}_{\text{Observing}D_{cb}^{SCP(n_c)}}.$$

From Lemma A.11, we know that $\beta_{-p,0}^{SCP(n_c)} = \sigma^2\beta_{-p,c}^{SCP(n_c)}$ and $\beta_{-c,0}^{SCP(n_c)} = \sigma^2\beta_{-c,c}^{SCP(n_c)}$. Thus:

$$\sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)} = \beta_{-p,0}^{SCP(n_c)} + (n_c-1)\beta_{-p,c}^{SCP(n_c)} = \left(\sigma^2+n_c-1\right)\beta_{-p,c}^{SCP(n_c)}$$
$$\sum_{j=0}^{n} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)} = \beta_{-p,0}^{SCP(n_c)} + (n_c)\beta_{-p,c}^{SCP(n_c)} = \left(\sigma^2+n_c\right)\beta_{-p,c}^{SCP(n_c)}$$
$$\sum_{j\notin D_{cb}^{SCP(n_c)}} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)} = \beta_{-c,0}^{SCP(n_c-)} + (n_c-1)\beta_{-c,c}^{SCP(n_c)} = \left(\sigma^2+n_c-1\right)\beta_{-c,c}^{SCP(n_c)}$$

and

$$\sum_{j=0}^{1} g_{c,j}^{SCP(n_c)}\beta_{-c,j}^{SCP(n_c)} = \beta_{-c,0}^{SCP(n_c-)} + (n_c)\beta_{-c,c}^{SCP(n_c)} = \left(\sigma^2+n_c-\right)\beta_{-c,c}^{SCP(n_c)}$$

Hence, we can write the payoff loss of a core player braking a lonk as follows:

$$-\Delta U_{cb}^{SCP}(n_c) = \frac{\sigma^2\left(1-r\left(\sigma^2+n_c-1\right)\beta_{-c,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c-1} - \frac{\sigma^2\left(1-r\left(\sigma^2+n_c\right)\beta_{-c,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c} + r^2\sigma^2\beta_{-c,c}^{SCP(n_c)2}$$
$$-\Delta U_{pb}^{SCP}(n_c) = \frac{\sigma^2\left(1-r\left(\sigma^2+n_c-1\right)\beta_{-p,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c} - \frac{\sigma^2\left(1-r\left(\sigma^2+n_c\right)\beta_{-p,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c+1} + r^2\sigma^2\beta_{-p,c}^{SCP(n_c)2}$$

We can simplify $-\Delta U_{cb}^{SCP}(n_c)$ as follows:

$$-\Delta U_{cb}^{SCP}(n_c) = \frac{\sigma^2\left(1-r\left(\sigma^2+n_c-1\right)\beta_{-c,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c-1} - \frac{\sigma^2\left(1-r\left(\sigma^2+n_c\right)\beta_{-c,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c} + r^2\sigma^2\beta_{-c,c}^{SCP(n_c)2}$$

$$
\begin{aligned}
&= \frac{\sigma^2}{\sigma^2+n_c-1}\left(1-2r(\sigma^2+n_c-1)\beta_{-c,c}^{SCP(n_c)}+r^2\beta_{-c,c}^{SCP(n_c)^2}(\sigma^2+n_c-1)^2\right) \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c}\left(1-2r(\sigma^2+n_c)\beta_{-c,c}^{SCP(n_c)}+r^2\beta_{-c,c}^{SCP(n_c)^2}(\sigma^2+n_c)^2\right)+r^2\sigma^2\beta_{-c,c}^{SCP(n_c)^2} \\
&= \frac{\sigma^2}{\sigma^2+n_c-1}-2\sigma^2 r\beta_{-c,c}^{SCP(n_c)}+\sigma^2 r^2\beta_{-c,c}^{SCP(n_c)^2}(\sigma^2+n_c-1) \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c}+2\sigma^2 r\beta_{-c,c}^{SCP(n_c)}-\sigma^2 r^2\beta_{-c,c}^{SCP(n_c)^2}(\sigma^2+n_c)+r^2\sigma^2\beta_{-c,c}^{SCP(n_c)^2} \\
&= \frac{\sigma^2}{\sigma^2+n_c-1}-\frac{\sigma^2}{\sigma^2+n_c} \\
&= \frac{\sigma^2}{(\sigma^2+n_c)(\sigma^2+n_c-1)}
\end{aligned}
$$

and we can simplify $-\Delta U_{pb}^{SCP}(n_c)$ as follows:

$$
\begin{aligned}
-\Delta U_{pb}^{SCP}(n_c) &= \frac{\sigma^2\left(1-r\left(\sigma^2+n_c-1\right)\beta_{-p,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c}-\frac{\sigma^2\left(1-r\left(\sigma^2+n_c\right)\beta_{-p,c}^{SCP(n_c)}\right)^2}{\sigma^2+n_c+1}+r^2\sigma^2\beta_{-p,c}^{SCP(n_c)^2} \\
&= \frac{\sigma^2}{\sigma^2+n_c}\left(1-r(\sigma^2+n_c)\beta_{-p,c}^{SCP(n_c)}+r\beta_{-p,c}^{SCP(n_c)}\right)^2 \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c+1}\left(1-r(\sigma^2+n_c+1)\beta_{-p,c}^{SCP(n_c)}+r\beta_{-p,c}^{SCP(n_c)}\right)^2+r^2\sigma^2\beta_{-p,c}^{SCP(n_c)^2} \\
&= \frac{\sigma^2}{\sigma^2+n_c}\left(1+r^2(\sigma^2+n_c)^2\beta_{-p,c}^{SCP(n_c)^2}+r^2\beta_{-p,c}^{SCP(n_c)^2}-2r(\sigma^2+n_c)\beta_{-p,c}^{SCP(n_c)}+2r\beta_{-p,c}^{SCP(n_c)}\right. \\
&\quad \left.-2r^2(\sigma^2+n_c)\beta_{-p,c}^{SCP(n_c)^2}\right) \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c+1}\left(1+r^2(\sigma^2+n_c+1)^2\beta_{-p,c}^{SCP(n_c)^2}+r^2\beta_{-p,c}^{SCP(n_c)^2}-2r(\sigma^2+n_c+1)\beta_{-p,c}^{SCP(n_c)}\right. \\
&\quad \left.+2r\beta_{-p,c}^{SCP(n_c)}-2r^2(\sigma^2+n_c+1)\beta_{-p,c}^{SCP(n_c)^2}\right)+r^2\sigma^2\beta_{-p,c}^{SCP(n_c)^2} \\
&= \frac{\sigma^2}{\sigma^2+n_c}+\sigma^2 r^2(\sigma^2+n_c)\beta_{-p,c}^{SCP(n_c)^2}+\frac{\sigma^2}{\sigma^2+n_c}r^2\beta_{-p,c}^{SCP(n_c)^2}-2\sigma^2 r\beta_{-p,c}^{SCP(n_c)}+\frac{\sigma^2}{\sigma^2+n_c}2r\beta_{-p,c}^{SCP(n_c)} \\
&\quad -2\sigma^2 r^2\beta_{-p,c}^{SCP(n_c)^2} \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c+1}-\sigma^2 r^2(\sigma^2+n_c+1)\beta_{-p,c}^{SCP(n_c)^2}-\frac{\sigma^2}{\sigma^2+n_c+1}r^2\beta_{-p,c}^{SCP(n_c)^2}+2\sigma^2 r\beta_{-p,c}^{SCP(n_c)} \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c+1}2r\beta_{-p,c}^{SCP(n_c)}+2\sigma^2 r^2\beta_{-p,c}^{SCP(n_c)^2}+r^2\sigma^2\beta_{-p,c}^{SCP(n_c)^2} \\
&= \frac{\sigma^2}{\sigma^2+n_c}-\frac{\sigma^2}{\sigma^2+n_c+1}+\frac{\sigma^2}{\sigma^2+n_c}r^2\beta_{-p,c}^{SCP(n_c)^2}+\frac{\sigma^2}{\sigma^2+n_c}2r\beta_{-p,c}^{SCP(n_c)} \\
&\quad - \frac{\sigma^2}{\sigma^2+n_c+1}r^2\beta_{-p,c}^{SCP(n_c)^2}-\frac{\sigma^2}{\sigma^2+n_c+1}2r\beta_{-p,c}^{SCP(n_c)} \\
&= \left[\frac{\sigma^2}{\sigma^2+n_c}-\frac{\sigma^2}{\sigma^2+n_c+1}\right]\left(1+2r\beta_{-p,c}^{SCP(n_c)}+r^2\beta_{-p,c}^{SCP(n_c)^2}\right) \\
&= \left[\frac{\sigma^2}{\sigma^2+n_c}-\frac{\sigma^2}{\sigma^2+n_c+1}\right]\left(1+r\beta_{-p,c}^{SCP(n_c)}\right)^2 \\
&= \frac{\sigma^2\left(1+r\beta_{-p,c}^{SCP(n_c)}\right)^2}{(\sigma^2+n_c)(\sigma^2+n_c+1)}
\end{aligned}
$$

From Lemma A.11 and using the fact that $r < 1$, we know that

$$r\beta_{-p,c}^{SCP(n_c)} \leq \frac{1}{\sigma^2 + n_c}.$$

Hence, we can write the payoff difference follows:

$$\begin{aligned}
-\Delta U_{cb}^{SCP}(n_c) + \Delta U_{pb}^{SCP}(n_c) &= \frac{\sigma^2}{(\sigma^2+n_c)(\sigma^2+n_c-1)} - \frac{\sigma^2\left(1+r\beta_{-p,c}^{SCP(n_c)}\right)^2}{(\sigma^2+n_c)(\sigma^2+n_c+1)} \\
&= \frac{\sigma^2}{(\sigma^2+n_c)}\left[\frac{1}{\sigma^2+n_c-1} - \frac{1}{\sigma^2+n_c+1}\left(1+r\beta_{-p,c}^{SCP(n_c)}\right)^2\right] \\
&\geq \frac{\sigma^2}{(\sigma^2+n_c)}\left[\frac{1}{\sigma^2+n_c-1} - \frac{1}{\sigma^2+n_c+1}\left(1+\frac{1}{\sigma^2+n_c}\right)^2\right] \\
&= \frac{\sigma^2}{(\sigma^2+n_c)}\left[\frac{1}{\sigma^2+n_c-1} - \frac{1}{(\sigma^2+n_c+1)}\frac{(\sigma^2+n_c+1)^2}{(\sigma^2+n_c)^2}\right] \\
&= \frac{\sigma^2}{(\sigma^2+n_c)}\left[\frac{1}{\sigma^2+n_c-1} - \frac{(\sigma^2+n_c+1)}{(\sigma^2+n_c)^2}\right] \\
&= \frac{\sigma^2(\sigma^2+n_c+1)}{(\sigma^2+n_c)}\left[\frac{1}{(\sigma^2+n_c-1)(\sigma^2+n_c+1)} - \frac{1}{(\sigma^2+n_c)^2}\right] \\
&= \frac{\sigma^2(\sigma^2+n_c+1)}{(\sigma^2+n_c)}\left[\frac{1}{(\sigma^2+n_c)^2-1} - \frac{1}{(\sigma^2+n_c)^2}\right] > 0.
\end{aligned}$$

□

In Lemma A.17, we fully characterize the necessary and sufficient conditions for an SPC network with n_c core players to be an equilibrium.

Lemma A.17. *A particular SCP network with $0 < n_c < n$ players in the core is an equilibrium if, and only if, $\Delta U_{cf}^{SCP}(n_c) \leq c_{n_c} \leq -\Delta U_{pb}^{SCP}(n_c)$. An empty network is an equilibrium if and only if $\Delta U_{cf}^{SCP}(n_c) \leq c_1$. A full network is an equilibrium if and only if $c_n \leq -\Delta U_{pb}^{SCP}(n_c)$.*

Proof. Let us focus on the sufficiency of the statement as necessity is trivial as those are equilibrium conditions.

By lemma A.14, we started with four deviations to consider. We have that an $SCP(n_c)$ network is an equilibrium if and only all of the four deviations decreases an agent's payoff. That is, if $\Delta U_{cb}^{SCP}(n_c) + c_{n_c-1} \leq 0$, $\Delta U_{pb}^{SCP}(n_c) + c_{n_c} \leq 0$, $\Delta U_{cf}^{SCP}(n_c) - c_{n_c} \leq 0$, and $\Delta U_{pf}^{SCP}(n_c) - c_{n_c+1} \leq 0$.

Using Result D and the fact that $c_{n_c} \leq c_{n_c+1}$, we have that $\Delta U_{cf}^{SCP}(n_c) - c_{n_c} \leq 0$ implies $\Delta U_{pf}^{SCP}(n_c) - c_{n_c+1} \leq 0$. The two conditions can be subsumed by $\Delta U_{cf}^{SCP}(n_c) \leq c_{n_c}$.

Similarly, using Result E and the fact that $c_{n_c-1} \leq c_{n_c}$, we have that $\Delta U_{pb}^{SCP}(n_c) + c_{n_c} \leq 0$ implies $\Delta U_{cb}^{SCP}(n_c) + c_{n_c-1} \leq 0$. The two conditions can be subsumed by $c_{n_c} \leq -\Delta U_{pb}^{SCP}(n_c)$.

□

Thus, to check whether a particular simple-core-periphery network with n_c members in the core is a pure strategy equilibrium, it is sufficient to only evaluate whether two possible deviations are profitable: (i) a core member forming a link, and (ii) a periphery member breaking a link. The identity of the particular agent is irrelevant. If such network is not an equilibrium it must be that either one of this two deviations is profitable.

CPOD networks Second, let us now consider a core-periphery-observing-down network with n_c core players, $CPOD(n_c)$. Agents in the core observe all other core agents' signals, and also one signal from a periphery agent. Thus not all periphery agents are alike, one of the periphery agents is special (we will call it the $n_c + 1$ agent). There are three different different types of agents: (i) an agent in the core, whose signal is observed by all other agents; (ii) a special periphery agent—the $n_c + 1$ agent—whose signal is observed by all core agents; and (iii) a normal periphery agent, whose signal is not observed by any other agent.

Using lemma A.14, we can restrict the set of deviations to six deviations relevant: an agent of either one of the three groups can break or form a new link. Let us formally define such deviations:

$$\Delta\Pi^{CPOD}(n_c) = \begin{cases} \Delta\Pi^{CPOD}_{cb}(n_c) & \text{core agent breaks a link} \\ \Delta\Pi^{CPOD}_{cf}(n_c) & \text{core agent forms a link} \\ \Delta\Pi^{CPOD}_{n_c+1b}(n_c) & \text{special periphery agent breaks a link} \\ \Delta\Pi^{CPOD}_{n_c+1f}(n_c) & \text{special periphery agent forms a link} \\ \Delta\Pi^{CPOD}_{pb}(n_c) & \text{periphery agent breaks a link} \\ \Delta\Pi^{CPOD}_{pf}(n_c) & \text{periphery agent forms a link} \end{cases}$$

$$\begin{aligned}
\Delta\Pi^{CPOD}_{cb}(n_c) &= (U_{cb}(n_c) - C(n_c - 1)) - (U_c(n_c) - C(n_c)) = \Delta U^{CPOD}_{cb}(n_c) + c_{n_c} \\
\Delta\Pi^{CPOD}_{cf}(n_c) &= (U_{cf}(n_c) - C(n_c + 1)) - (U_c(n_c) - C(n_c)) = \Delta U^{CPOD}_{cf}(n_c) - c_{n_c+1} \\
\Delta\Pi^{CPOD}_{n_c+1b}(n_c) &= (U_{n_c+1b}(n_c) - C(n_c - 1)) - (U_{n_c+1}(n_c) - C(n_c)) = \Delta U^{CPOD}_{n_c+1b}(n_c) + c_{n_c} \\
\Delta\Pi^{CPOD}_{n_c+1f}(n_c) &= (U_{n_c+1f}(n_c) - C(n_c + 1)) - (U_{n_c+1}(n_c) - C(n_c)) = \Delta U^{CPOD}_{n_c+1f}(n_c) - c_{n_c+1} \\
\Delta\Pi^{CPOD}_{pb}(n_c) &= (U_{pb}(n_c) - C(n_c - 1)) - (U_p(n_c) - C(n_c)) = \Delta U^{CPOD}_{pb}(n_c) + c_{n_c} \\
\Delta\Pi^{CPOD}_{pf}(n_c) &= (U_{pf}(n_c) - C(n_c + 1)) - (U_p(n_c) - C(n_c)) = \Delta U^{CPOD}_{pf}(n_c) - c_{n_c+1}
\end{aligned}$$

In order to begin limiting the possible deviations, note that by Lemma A.3, if a normal periphery agent would form a new link, she would always choose to observe the special periphery agent, and not another periphery agent. Also by Lemma A.3, if a core player were to break a link, it would stop observing the signal of the special periphery agent.

Note that all agents observe the same number of signals in equilibrium. Also, if a core player would stop observing a signal, she would miss on the less influential $n_c + 1$ agent's signal, while if either type of periphery agent were to stop observing a signal, they would miss on a more influential core signal. Intuitively, this gives us that a core agent has less to lose by breaking a link than a periphery agent of either type.

Lemma A.18. *For any CPOD network with n_c agents in the core, $CPOD(n_c)$, we have that $\Delta U^{CPOD}_{cb}(n_c) \geq \Delta U^{CPOD}_{n_c+1b}(n_c)$.*

Proof. To show item (a), note that both the core agent and $n_c + 1$ observe the same signals, and thus have the same information set $g_{cj} = g_{n_c+1j}\ \forall j$. By Lemma A.12, we have that $\beta^{CPOD}_c(n_c) \geq \beta^{CPOD}_{n_c+1}(n_c)$. By Lemma A.4, the inequality is preserved when we look at β_{-i} for any i. By Lemma A.3, we can compare the core agent deviation of breaking the link with $n_c + 1$ player, and a fictitious deviation of breaking the link with himself. We have that if the core agent breaks a link she stops observing the $n_c + 1$ agent's signal, and looses $\beta_{-in_c+1} \leq \beta_{-ic}$. □

Lemma A.19. *For any core-periphery-observing-down network with $n_c - 1$ core players, $CPOD(n_c - 1)$, we have $\Delta U^{CPOD}_{pb}(n_c - 1) < \Delta U^{CPOD}_{cb}(n_c - 1)$ for every $n_c = 2, \ldots, n$.*

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U_{cb}^{CPOD}(n_c-1) = -\underbrace{\frac{\sigma^2\left(1-r\sum_{j\notin D_{cb}^{CPOD(n_c)}} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)}\right)^2}{\sigma^2+n_c-1}}_{\text{Without observing}}$$

$$+\underbrace{\frac{\sigma^2\left(1-r\sum_{j=0}^{n} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)}\right)^2}{\sigma^2+n_c}}_{\text{Observing} D_{cb}^{CPOD(n_c)}}$$

$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{CPOD(n_c)})\beta_{-c,j}^{CPOD(n_c)2}+\sum_{j\in D_{cb}^{CPOD(n_c)}}\beta_{-c,j}^{CPOD(n_c)2}\right)}_{\text{Without observing}}$$

$$\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{c,j}^{CPOD(n_c)})\beta_{-c,j}^{CPOD(n_c)2}}_{\text{Observing} D_{cb}^{CPOD(n_c)}}$$

$$= -\left(1-r\sum_{j\notin D_{cb}^{CPOD(n_c)}} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c-1}-\frac{\sigma^2}{\sigma^2+n_c}\right)$$

$$-2\frac{r\sigma^2}{\sigma^2+n_c}\left(1-r\sum_{j\notin D_{cb}^{CPOD(n_c)}} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)}\right)\beta_{-c,n_c}^{CPOD(n_c)}$$

$$-r^2\sigma^2\beta_{-c,n_c}^{CPOD(n_c)2}\left[1-\frac{1}{\sigma^2+n_c}\right],$$

and

$$\Delta U_{pb}^{CPOD}(n_c-1) = -\underbrace{\frac{\sigma^2\left(1-r\sum_{j\notin D_{pb}^{CPOD(n_c)}} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)}\right)^2}{\sigma^2+n_c-1}}_{\text{Without observing}}$$

$$+\underbrace{\frac{\sigma^2\left(1-r\sum_{j=0}^{n} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)}\right)^2}{\sigma^2+n_c}}_{\text{Observing} D_{pb}^{CPOD(n_c)}}$$

$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{p,j}^{CPOD(n_c)})\beta_{-p,j}^{CPOD(n_c)2}+\sum_{j\in D_{pb}^{CPOD(n_c)}}\beta_{-p,j}^{CPOD(n_c)2}\right)}_{\text{Without observing}}$$

$$\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{p,j}^{CPOD(n_c)})\beta_{-p,j}^{CPOD(n_c)2}}_{\text{Observing} D_{pb}^{CPOD(n_c)}}$$

$$= -\left(1-r\sum_{j\notin D_{pb}^{CPOD(n_c)}} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c-1}-\frac{\sigma^2}{\sigma^2+n_c}\right)$$

$$- 2\frac{r\sigma^2}{\sigma^2 + n_c}\left(1 - r\sum_{j\notin D_{pb}^{CPOD(n_c)}} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)}\right)\beta_{-p,c}^{CPOD(n_c)}$$
$$- r^2\sigma^2\beta_{-p,c}^{CPOD(n_c)2}\left[1 - \frac{1}{\sigma^2 + n_c}\right],$$

Hence, to prove that $-\Delta U_{cb}^{CPOD}(n_c - 1) < -\Delta U_{pb}^{CPOD}(n_c - 1)$ it is sufficient to show that:

(i) $\sum_{j\notin D_{pb}^{CPOD(n_c)}} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)} < \sum_{j\notin D_{cb}^{CPOD(n_c)}} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)}$, and

(ii) $\beta_{-c,n_c}^{CPOD(n_c)} < \beta_{-p,c}^{CPOD(n_c)}$.

Notice that:

$$\sum_{j\notin D_{cb}^{CPOD(n_c)}} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)} = \beta_{-c,0}^{CPOD(n_c)} + (n_c - 1)\beta_{-c,c}^{CPOD(n_c)} = 1 - \beta_{-c,n_c}^{CPOD(n_c)} - (n - n_c)\beta_{-c,p}^{CPOD(n_c)}$$
$$\sum_{j\notin D_{pb}^{CPOD(n_c)}} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)} = \beta_{-p,0}^{CPOD(n_c)} + (n_c - 2)\beta_{-p,c}^{CPOD(n_c)} = 1 - \beta_{-p,c}^{CPOD(n_c)} - \beta_{-p,n_c}^{CPOD(n_c)} - (n - n_c - 1)\beta_{-p,p}^{CPOD(n_c)}$$

Thus:

$$\sum_{j\notin D_{cb}^{CPOD(n_c)}} g_{c,j}^{CPOD(n_c)}\beta_{-c,j}^{CPOD(n_c)} - \sum_{j\notin D_{pb}^{CPOD(n_c)}} g_{p,j}^{CPOD(n_c)}\beta_{-p,j}^{CPOD(n_c)}$$
$$= \left[1 - \beta_{-c,n_c}^{CPOD(n_c)} - (n - n_c)\beta_{-c,p}^{CPOD(n_c)}\right] - \left[1 - \beta_{-p,c}^{CPOD(n_c)} - \beta_{-p,n_c}^{CPOD(n_c)} - (n - n_c - 1)\beta_{-p,p}^{CPOD(n_c)}\right]$$
$$= \underbrace{\beta_{-p,c}^{CPOD(n_c)} - \beta_{-c,n_c}^{CPOD(n_c)}}_{>0 \text{ (Lemma A.12)}} + \underbrace{\beta_{-p,n_c}^{CPOD(n_c)} - \beta_{-p,p}^{CPOD(n_c)}}_{>0 \text{ (Lemma A.12)}} > 0.$$

Furthermore, from Lemma A.12, we know that $\beta_{-c,n_c}^{CPOD(n_c)} < \beta_{-p,c}^{CPOD(n_c)}$, which guarantees the second sufficient conditions mentioned above, which guarantees the second sufficient conditions mentioned above. □

Also, note that an agent in the core or a special periphery agent have the same information set (not only the same number of signals observed) and thus the same equilibrium expected payoff. Furthermore, if they were to observe an extra signal, they would both equally profit from it. Thus, for any number of agents in the core n_c, we have that:

$$\Delta U_{n_c+1f}^{CPOD}(n_c) = \Delta U_{cf}^{CPOD}(n_c)$$

Finally, a core agent is more informed than a normal periphery agent. By concavity, if a core agent observes an extra signal she has less to gain than if a periphery agent were to do so.

Lemma A.20. *For any core-periphery-observing-down network with* $n_c - 1$ *core players,* $CPOD(n_c - 1)$*, we have* $\Delta U_{cf}^{CPOD}(n_c - 1) < \Delta U_{pf}^{CPOD}(n_c - 1)$ *for every* $n_c = 2, \ldots, n$.

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U_{pf}^{CPOD}(n_c - 1) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} - r\sum_{j\in D_{pf}^{CPOD(n_c-1)}}\beta_{-p,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2 + n_c + 1}}_{\text{Observing } D_{pf}^{CPOD(n_c-1)}}$$

$$
\begin{aligned}
&+ \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}} \\
&- \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{CPOD(n_c-1)})\beta_{-p,j}^{CPOD(n_c-1)2} - \sum_{j\in D_{pf}^{CPOD(n_c-1)}}\beta_{-p,j}^{CPOD(n_c-1)2}\right)}_{\text{Observing } D_{pf}^{CPOD(n_c-1)}} \\
&+ \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{CPOD(n_c-1)})\beta_{-p,j}^{CPOD(n_c-1)2}\right)}_{\text{Without observing}} \\
= &\left(1 - r\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c} - \frac{\sigma^2}{\sigma^2+n_c+1}\right) \\
&+ 2\frac{r\sigma^2}{\sigma^2+n_c+1}\left(1 - r\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}\right)\beta_{-p,n_c}^{CPOD(n_c-1)} \\
&+ r^2\sigma^2\beta_{-p,n_c}^{CPOD(n_c-1)2}\left[1 - \frac{1}{\sigma^2+n_c+1}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta U_{cf}^{CPOD}(n_c-1) = &- \underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{cf}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)} - r\sum_{j\in D_{cf}^{CPOD(n_c-1)}}\beta_{-c,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2+n_c+1}}_{\text{Observing } D_{cf}^{CPOD(n_c-1)}} \\
&+ \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}} \\
&- \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{c,j}^{CPOD(n_c-1)})\beta_{-c,j}^{CPOD(n_c-1)2} - \sum_{j\in D_{cf}^{CPOD(n_c-1)}}\beta_{-c,j}^{CPOD(n_c-1)2}\right)}_{\text{Observing } D_{cf}^{CPOD(n_c-1)}} \\
&+ \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{c,j}^{CPOD(n_c-1)})\beta_{-c,j}^{CPOD(n_c-1)2}\right)}_{\text{Without observing}} \\
= &\left(1 - r\sum_{j\notin D_{cf}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c} - \frac{\sigma^2}{\sigma^2+n_c+1}\right) \\
&+ 2\frac{r\sigma^2}{\sigma^2+n_c+1}\left(1 - r\sum_{j\notin D_{cf}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)\beta_{-c,p}^{CPOD(n_c-1)} \\
&+ r^2\sigma^2\beta_{-c,p}^{CPOD(n_c-1)2}\left[1 - \frac{1}{\sigma^2+n_c+1}\right]
\end{aligned}
$$

Hence, to prove that $\Delta U_{cf}^{CPOD}(n_c-1) < \Delta U_{pf}^{CPOD}(n_c-1)$ it is sufficient to show that:

(i) $\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} < \sum_{j\notin D_{cf}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}$, and

(ii) $\beta_{-c,p}^{CPOD(n_c-1)} < \beta_{-p,n_c}^{CPOD(n_c-1)}$.

Notice that:

$$\begin{aligned}\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} &= \beta_{-p,0}^{CPOD(n_c-1)} + (n_c-1)\beta_{-p,c}^{CPOD(n_c-1)} \\ &= 1-\beta_{-p,n_c}^{CPOD(n_c-1)} - (n-n_c-1)\beta_{-p,p}^{CPOD(n_c-1)}\end{aligned}$$

and

$$\begin{aligned}\sum_{j\notin D_{cf}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)} &= \beta_{-c,0}^{CPOD(n_c-1)} + (n_c-1)\beta_{-c,c}^{CPOD(n_c-1)} + \beta_{-c,n_c}^{CPOD(n_c-1)} \\ &= 1-(n-n_c)\beta_{-c,p}^{CPOD(n_c-1)}\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{j\notin D_{cf}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)} - \sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} \\ &= (n-n_c)\underbrace{\left[\beta_{-p,p}^{CPOD(n_c-1)} - \beta_{-c,p}^{CPOD(n_c-1)}\right]}_{=0\text{ (Lemma A.12)}} + \underbrace{\beta_{-c,n_c}^{CPOD(n_c-1)} - \beta_{-p,p}^{CPOD(n_c-1)}}_{>0\text{ (Lemma A.12)}} > 0.\end{aligned}$$

Furthermore, from Lemma A.12, we know that $\beta_{-c,p}^{CPOD(n_c-1)} < \beta_{-p,n_c}^{CPOD(n_c-1)}$, which guarantees the second sufficient conditions mentioned above. □

Lemma A.21. *A particular CPOD network with $0 < n_c < n$ players in the core is an equilibrium if, and only if, $\Delta U_{PF}^{CPOD}(n_c) \leq c_{n_c+1}$ and $-\Delta U_{CP}^{CPOD}(n_c) \geq c_{n_c}$.*

Proof. Let us focus on the sufficiency of the statement, as the necessity is trivial.

By lemma A.14, we start with six deviations. A $CPOD(n_c)$ network is an equilibrium if and only if all six deviations are not profitable for the deviating agent. That is, if $\Delta U_{cb}^{CPOD}(n_c) + c_{n_c} \leq 0$, $\Delta U_{cf}^{CPOD}(n_c) - c_{n_c+1} \leq 0$, $\Delta U_{n_c+1b}^{CPOD}(n_c) + c_{n_c} \leq 0$, $\Delta U_{n_c+1f}^{CPOD}(n_c) - c_{n_c+1} \leq 0$, $\Delta U_{pb}^{CPOD}(n_c) + c_{n_c} \leq 0$, and $\Delta U_{pf}^{CPOD}(n_c) - c_{n_c+1} \leq 0$.

Using Result A, we have that $\Delta U_{cb}^{CPOD}(n_c) + c_{n_c} \leq 0$ implies both $\Delta U_{n_c+1b}^{CPOD}(n_c) + c_{n_c} \leq 0$ and $\Delta U_{pb}^{CPOD}(n_c) + c_{n_c} \leq 0$. $\Delta U_{cb}^{CPOD}(n_c) + c_{n_c} \leq 0$ can be rewritten as $-\Delta U_{cb}^{CPOD}(n_c) \geq c_{n_c}$.

Using Result C, we have that $\Delta U_{pf}^{CPOD}(n_c) - c_{n_c+1} \leq 0$ implies $\Delta U_{cf}^{CPOD}(n_c) - c_{n_c+1} \leq 0$, which is equivalent to $\Delta U_{n_c+1f}^{CPOD}(n_c) - c_{n_c+1} \leq 0$. Thus a sufficient condition is $\Delta U_{pf}^{CPOD}(n_c) \leq c_{n_c+1}$ □

To check whether a particular core-periphery-observing-down network with n_c members in the core is a pure strategy equilibrium, it is sufficient to only evaluate whether two possible deviations are profitable: (i) a core member breaking a link, and (ii) a periphery member forming a link. If such network is not an equilibrium it must be that either one of this two deviations is profitable.

C.3 Evaluating the Deviations

Our work in the previous section shows us that for a particular core-periphery network to not be an equilibrium, it must be that either one of two conditions fail, and thus a deviation would be profitable.

Before proceeding to the main theorem and proof, we next establish relationships between these deviations. First, we show that a periphery agent has less to gain by deviating and forming the n_c^{th} link from a $CPOD(n_c - 1)$ network, than what she looses if she breaks the n_c^{th} link from a $SCP(n_c)$ network. That is:

Lemma A.22. *For any simple-core-periphery network with n_c core players, $SCP(n_c)$, and any core-periphery-observing-down network with $n_c - 1$ core players, $CPOD(n_c - 1)$, we have $\Delta U_{pf}^{CPOD}(n_c - 1) < -\Delta U_{pb}^{SCP}(n_c)$ for every $n_c = 1, \ldots, n$.*

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U_{pb}^{SCP}(n_c) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)^2}{\sigma^2 + n_c + 1}}_{\text{Observing} D_{pb}^{SCP(n_c)}}$$

$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{SCP(n_c)})\beta_{-p,j}^{SCP(n_c)2} + \sum_{j\in D_{pb}^{SCP(n_c)}}\beta_{-p,j}^{SCP(n_c)2}\right)}_{\text{Without observing}}$$

$$\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1 - g_{p,j}^{SCP(n_c)})\beta_{-p,j}^{SCP(n_c)2}}_{\text{Observing} D_{pb}^{SCP(n_c)}}$$

$$= -\left(1 - r\sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2 + n_c} - \frac{\sigma^2}{\sigma^2 + n_c + 1}\right)$$

$$- 2\frac{r\sigma^2}{\sigma^2 + n_c + 1}\left(1 - r\sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\right)\beta_{-p,c}^{SCP(n_c)}$$

$$- r^2\sigma^2\beta_{-p,c}^{SCP(n_c)2}\left[1 - \frac{1}{\sigma^2 + n_c + 1}\right],$$

and

$$\Delta U_{pf}^{CPOD}(n_c - 1) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} - r\sum_{j\in D_{pf}^{CPOD(n_c-1)}}\beta_{-p,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2 + n_c + 1}}_{\text{Observing } D_{pf}^{CPOD(n_c-1)}}$$

$$+ \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}}$$

$$\underbrace{- r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{CPOD(n_c-1)})\beta_{-p,j}^{CPOD(n_c-1)2} - \sum_{j\in D_{pf}^{CPOD(n_c-1)}}\beta_{-p,j}^{CPOD(n_c-1)2}\right)}_{\text{Observing } D_{pf}^{CPOD(n_c-1)}}$$

$$
+ \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{p,j}^{CPOD(n_c-1)})\beta_{-p,j}^{CPOD(n_c-1)2}\right)}_{\text{Without observing}}
$$

$$
\begin{aligned}
&= \left(1 - r\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c}-\frac{\sigma^2}{\sigma^2+n_c+1}\right)\\
&\quad + 2\frac{r\sigma^2}{\sigma^2+n_c+1}\left(1-r\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}\right)\beta_{-p,n_c}^{CPOD(n_c-1)}\\
&\quad + r^2\sigma^2\beta_{-p,n_c}^{CPOD(n_c-1)2}\left[1-\frac{1}{\sigma^2+n_c+1}\right]
\end{aligned}
$$

Hence, to prove that $\Delta U_{pf}^{CPOD}(n_c-1) < -\Delta U_{pb}^{SCP}(n_c)$ it is sufficient to show that:

(i) $\sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)} < \sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)}$, and

(ii) $\beta_{-p,n_c}^{CPOD(n_c-1)} < \beta_{-p,c}^{SCP(n_c)}$.

Notice that:

$$
\sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)} = \beta_{-p,0}^{SCP(n_c)} + (n_c-1)\beta_{-p,c}^{SCP(n_c)}
$$

and

$$
\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} = \beta_{-p,0}^{CPOD(n_c-1)} + n_c\beta_{-p,c}^{CPOD(n_c-1)}
$$

Thus,

$$
\begin{aligned}
&\sum_{j\notin D_{pf}^{CPOD(n_c-1)}} g_{p,j}^{CPOD(n_c-1)}\beta_{-p,j}^{CPOD(n_c-1)} - \sum_{j\notin D_{pb}^{SCP(n_c)}} g_{p,j}^{SCP(n_c)}\beta_{-p,j}^{SCP(n_c)}\\
&= \underbrace{\beta_{-p,c}^{CPOD(n_c-1)}}_{>0} + \underbrace{\beta_{-p,0}^{CPOD(n_c-1)} - \beta_{-p,0}^{SCP(n_c)}}_{>0\text{ (Lemma A.13)}} + (n_c-1)\underbrace{\left[\beta_{-p,c}^{CPOD(n_c-1)} - \beta_{-p,c}^{SCP(n_c)}\right]}_{>0\text{ (Lemma A.13)}} > 0.
\end{aligned}
$$

Furthermore, from Lemma A.13, we know that $\beta_{-p,n_c}^{CPOD(n_c-1)} < \beta_{-p,c}^{SCP(n_c)}$, which guarantees the second sufficient conditions mentioned above. □

Next, we compare different deviations a core agent can make in different networks. We show that for a given number of agents in the core, a particular agent in the core gains less from observing an extra signal in an $SCP(n_c)$ network than she loses if she stops observing a signal in a $CPOD(n_c)$ network.

Lemma A.23. *For any simple-core-periphery network with n_c-1 core players, $SCP(n_c-1)$, and any core-periphery-observing-down network with n_c-1 core players, $CPOD(n_c-1)$, we have $\Delta U_{cf}^{SCP}(n_c-1) < -\Delta U_{cb}^{CPOD}(n_c-1)$ for every $n_c = 2,\ldots,n$.*

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U_{cb}^{CPOD}(n_c) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{cb}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2+n_c-1}}_{\text{Without observing}}$$

$$+\underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)^2}{\sigma^2+n_c}}_{\text{Observing}D_{cb}^{CPOD(n_c-1)}}$$

$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{CPOD(n_c-1)})\beta_{-c,j}^{CPOD(n_c-1)2} + \sum_{j\in D_{cb}^{CPOD(n_c-1)}}\beta_{-c,j}^{CPOD(n_c-1)2}\right)}_{\text{Without observing}}$$

$$\underbrace{+r^2\sigma^2\sum_{j=0}^{n}(1-g_{c,j}^{CPOD(n_c-1)})\beta_{-c,j}^{CPOD(n_c-1)2}}_{\text{Observing}D_{cb}^{CPOD(n_c-1)}}$$

$$= -\left(1 - r\sum_{j\notin D_{cb}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c-1} - \frac{\sigma^2}{\sigma^2+n_c}\right)$$

$$- 2\frac{r\sigma^2}{\sigma^2+n_c}\left(1 - r\sum_{j\notin D_{cb}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}\right)\beta_{-c,n_c}^{CPOD(n_c-1)}$$

$$- r^2\sigma^2\beta_{-c,n_c}^{CPOD(n_c-1)2}\left[1 - \frac{1}{\sigma^2+n_c}\right],$$

and

$$\Delta U_{cf}^{SPC}(n_c-1) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{cf}^{SCP(n_c-1)}} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)} - r\sum_{j\in D_{cf}^{SCP(n_c-1)}}\beta_{-c,j}^{SCP(n_c-1)}\right)^2}{\sigma^2+n_c}}_{\text{Observing } D_{cf}^{SCP(n_c-1)}}$$

$$+\underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)}\right)^2}{\sigma^2+n_c-1}}_{\text{Without observing}}$$

$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{SCP(n_c-1)})\beta_{-c,j}^{SCP(n_c-1)2} - \sum_{j\in D_{cf}^{SCP(n_c-1)}}\beta_{-c,j}^{SCP(n_c-1)2}\right)}_{\text{Observing } D_{cf}^{SCP(n_c-1)}}$$

$$\underbrace{+r^2\sigma^2\left(\sum_{j=0}^{n}(1-g_{c,j}^{SCP(n_c-1)})\beta_{-c,j}^{SCP(n_c-1)2}\right)}_{\text{Without observing}}$$

$$=\left(1 - r\sum_{j\notin D_{cf}^{SCP(n_c-1)}} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)}\right)^2\left(\frac{\sigma^2}{\sigma^2+n_c-1} - \frac{\sigma^2}{\sigma^2+n_c}\right)$$

$$+ 2\frac{r\sigma^2}{\sigma^2 + n_c}\left(1 - r\sum_{j\notin D_{cf}^{SCP(n_c-1)}} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)}\right)\beta_{-c,p}^{SPC(n_c-1)}$$
$$+ r^2\sigma^2\beta_{-c,p}^{SPC(n_c-1)2}\left[1 - \frac{1}{\sigma^2 + n_c}\right]$$

Hence, to prove that $\Delta U_{cf}^{SCP}(n_c - 1) < -\Delta U_{cb}^{CPOD}(n_c - 1)$ it is sufficient to show that:

(i) $\sum_{j\notin D_{cb}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)} < \sum_{j\notin D_{cf}^{SCP(n_c-1)}} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)}$, and

(ii) $\beta_{-c,p}^{SPC(n_c-1)} < \beta_{-c,n_c}^{CPOD(n_c-1)}$.

Notice that:

$$\sum_{j\notin D_{cb}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)} = \beta_{-c,0}^{CPOD(n_c-1)} + (n_c - 1)\beta_{-c,c}^{CPOD(n_c-1)}$$

and

$$\sum_{j\notin D_{cf}^{SCP(n_c-1)}} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)} = \beta_{-c,0}^{SCP(n_c-1)} + (n_c - 1)\beta_{-c,c}^{SCP(n_c-1)}$$

Thus,

$$\sum_{j\notin D_{cf}^{SCP(n_c-1)}} g_{c,j}^{SCP(n_c-1)}\beta_{-c,j}^{SCP(n_c-1)} - \sum_{j\notin D_{cb}^{CPOD(n_c-1)}} g_{c,j}^{CPOD(n_c-1)}\beta_{-c,j}^{CPOD(n_c-1)}$$
$$= \underbrace{\beta_{-c,0}^{SCP(n_c-1)} - \beta_{-c,0}^{CPOD(n_c-1)}}_{>0 \text{ (Lemma A.13)}} + (n_c - 1)\underbrace{\left[\beta_{-c,c}^{SCP(n_c-1)} - \beta_{-c,c}^{CPOD(n_c-1)}\right]}_{>0 \text{ (Lemma A.13)}} > 0.$$

Furthermore, from Lemma A.13, we know that $\beta_{-c,p}^{SPC(n_c-1)} < \beta_{-c,n_c}^{CPOD(n_c-1)}$, which guarantees the second sufficient conditions mentioned above. □

Finally, we can compare the empty network with a network with one player in the center. We will use this in the first step of our induction.

Lemma A.24. *For a simple-core-periphery network with* 0 *core players,* $SCP(0)$*, and a simple-core-periphery network with* 1 *core player,* $SCP(1)$*, we have* $\Delta U_{pf}^{SCP}(0) < -\Delta U_{pb}^{SCP}(1)$.

Proof. Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U_{pb}^{SCP}(1) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{pb}^{SCP(1)}} g_{p,j}^{SCP(1)}\beta_{-p,j}^{SCP(1)}\right)^2}{\sigma^2 + 1}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{p,j}^{SCP(1)}\beta_{-p,j}^{SCP(1)}\right)^2}{\sigma^2 + 2}}_{\text{Observing} D_{pb}^{SCP(1)}}$$
$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{SCP(1)})\beta_{-p,j}^{SCP(1)2} + \sum_{j\in D_{pb}^{SCP(1)}}\beta_{-p,j}^{SCP(1)2}\right)}_{\text{Without observing}} + \underbrace{r^2\sigma^2\sum_{j=0}^{n}(1 - g_{p,j}^{SCP(1)})\beta_{-p,j}^{SCP(1)2}}_{\text{Observing} D_{pb}^{SCP(1)}}$$

$$= -\left(1 - r\sum_{j\notin D_{pb}^{SCP(1)}} g_{p,j}^{SCP(1)}\beta_{-p,j}^{SCP(1)}\right)^2\left(\frac{\sigma^2}{\sigma^2+1} - \frac{\sigma^2}{\sigma^2+2}\right)$$

$$- 2\frac{r\sigma^2}{\sigma^2+2}\left(1 - r\sum_{j\notin D_{pb}^{SCP(1)}} g_{p,j}^{SCP(1)}\beta_{-p,j}^{SCP(1)}\right)\beta_{-p,c}^{SCP(1)}$$

$$- r^2\sigma^2\beta_{-p,c}^{SCP(1)2}\left[1 - \frac{1}{\sigma^2+2}\right],$$

and

$$\Delta U_{pf}^{SPC}(0) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{pf}^{SCP(0)}} g_{p,j}^{SCP(0)}\beta_{-p,j}^{SCP(0)} - r\sum_{j\in D_{pf}^{SCP(0)}}\beta_{-p,j}^{SCP(0)}\right)^2}{\sigma^2+2}}_{\text{Observing } D_{pf}^{SCP(0)}}$$

$$+ \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{p,j}^{SCP(0)}\beta_{-p,j}^{SCP(0)}\right)^2}{\sigma^2+1}}_{\text{Without observing}}$$

$$- \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{SCP(0)})\beta_{-p,j}^{SCP(0)2} - \sum_{j\in D_{pf}^{SCP(0)}}\beta_{-p,j}^{SCP(0)2}\right)}_{\text{Observing } D_{pf}^{SCP(0)}}$$

$$+ \underbrace{r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j}^{SCP(0)})\beta_{-p,j}^{SCP(0)2}\right)}_{\text{Without observing}}$$

$$= \left(1 - r\sum_{j\notin D_{pf}^{SCP(0)}} g_{p,j}^{SCP(0)}\beta_{-p,j}^{SCP(0)}\right)^2\left(\frac{\sigma^2}{\sigma^2+1} - \frac{\sigma^2}{\sigma^2+2}\right)$$

$$+ 2\frac{r\sigma^2}{\sigma^2+2}\left(1 - r\sum_{j\notin D_{pf}^{SCP(0)}} g_{p,j}^{SCP(0)}\beta_{-p,j}^{SCP(0)}\right)\beta_{-p,p}^{SPC(0)}$$

$$+ r^2\sigma^2\beta_{-p,p}^{SPC(0)2}\left[1 - \frac{1}{\sigma^2+2}\right]$$

Hence, to prove that $\Delta U_{pf}^{SCP}(0) < -\Delta U_{pb}^{SCP}(1)$ it is sufficient to show that:

(i) $\sum_{j\notin D_{pb}^{SCP(1)}} g_{p,j}^{SCP(1)}\beta_{-p,j}^{SCP(1)} < \sum_{j\notin D_{pf}^{SCP(0)}} g_{p,j}^{SCP(0)}\beta_{-p,j}^{SCP(0)}$, and

(ii) $\beta_{-p,p}^{SPC(0)} < \beta_{-p,c}^{SCP(1)}$.

Notice that:

$$\sum_{j\notin D_{pb}^{SCP(1)}} g_{p,j}^{SCP(1)}\beta_{-p,j}^{SCP(1)} = \beta_{-p,0}^{SCP(1)}$$

and

$$\sum_{j \notin D_{pf}^{SCP(0)}} g_{p,j}^{SCP(0)} \beta_{-p,j}^{SCP(0)} = \beta_{-p,0}^{SCP(0)}$$

Thus,

$$\begin{aligned}
&\sum_{j \notin D_{pf}^{SCP(0)}} g_{p,j}^{SCP(0)} \beta_{-p,j}^{SCP(0)} - \sum_{j \notin D_{pb}^{SCP(1)}} g_{p,j}^{SCP(1)} \beta_{-p,j}^{SCP(1)} \\
&= \underbrace{\beta_{-p,0}^{SCP(0)} - \beta_{-p,0}^{SCP(1)}}_{>0 \text{ (Lemma A.11)}} > 0.
\end{aligned}$$

Furthermore, from Lemma A.11, we know that $\beta_{-p,p}^{SPC(0)} < \beta_{-p,c}^{SCP(1)}$, which guarantees the second sufficient conditions mentioned above. □

We are now finally ready to state and prove the main theorem of this section. We show that for any particular cost function that satisfies Assumption 2, either a SCP or a $CPOD$ type of network is an equilibrium.

Proposition A.1. *For any cost function $C(\cdot)$ satisfying Assumption 2, there exist at least one pure strategy linear action equilibrium. Furthermore, either a SCP or a $CPOD$ network is an equilibrium.*

Proof. Assume not; assume there is no equilibrium. We will proceed by induction on the number of players in the core of the equilibrium candidate network.

Since there is no equilibrium, we know that the empty network $SCP(0)$ is not an equilibrium. Thus, it must be that given the cost of acquiring information an agent is willing to deviate and form a connection, $c_1 < \Delta U_{pf}^{SCP}(0)$.

By Lemma A.24, this implies that $c_1 < -\Delta U_{pb}^{SCP}(1)$. Thus, for an $SCP(1)$ network not to be an equilibrium, it must be that the other condition is violated, and thus $c_1 < -\Delta U_{cf}^{SCP}(1)$.

By Lemma A.23, this last inequality implies that $c_1 < -\Delta U_{cb}^{CPOD}(1)$. So, for a $CPOD(1)$ network to not be an equilibrium, it must be that the other condition is violated, and thus $c_2 < \Delta U_{pf}^{CPOD}(1)$.

By Lemma A.22, this last inequality implies that $c_2 < -\Delta U_{pb}^{SCP}(2)$. So, for a $SCP(2)$ network to not be an equilibrium, it must be that the other condition is violated, and thus $c_2 < \Delta U_{cf}^{SCP}(2)$.

Note that if we apply Lemmata A.23 and A.22 again in a similar fashion, we obtain that $c_3 < \Delta U_{cf}^{SCP}(3)$. Proceeding inductively, we can apply Lemmata A.23, A.22, A.23, etc... We soon conclude that $c_n < -\Delta U_{pb}^{SCP}(n)$, which guarantees that the complete network is an equilibrium. This concludes our proof, by providing the contradiction. □

C.4 Any core-periphery network is an equilibrium for some cost function

Finally, we turn our attention to a different question. Given a core-periphery network, we show that there exists a cost function satisfying Assumption 2 such that the particular network is an equilibrium.

First, let us show that for any number of agents in the core, n_c, in an $SCP(n_c)$ network, a particular agent in the core has less to gain by forming a link than a periphery agent has to lose by breaking a link.

Lemma A.25. *For any simple-core-periphery network with n_c core players, $SCP(n_c)$, we have $\Delta U_{cf}^{SCP}(n_c) < -\Delta U_{pb}^{SCP}(n_c)$ for every $n_c = 1, \dots, n-1$.*

Proof. Let agents $i = 1, \dots, n_c$ be core players, and agents $i = n_c + 1, \dots, n$ be the periphery players. Also, let D_{cf} be the singleton set containing the agent that a core player would connect to if deviating by forming one additional connection. Since any core player observes all core agents' signal, we have $D_{cf} = \{n_c + 1\}$. Similarly, let D_{pb} be the singleton set containing the agent that a periphery player would connect to if deviating by breaking one connection. Since any periphery player observes all core agents' signal, we have $D_{pb} = \{n_c\}$.

Using Proposition 2 payoff formula, we can write the deviation gains as follows:

$$\Delta U_{pb}^{SCP}(n_c) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{p,j}\beta_{-p,j}^{SCP(n_c)}\right)^2}{\sigma^2 + n_c + 1}}_{\text{Observing} D_{pb}}$$
$$\underbrace{-r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{p,j})\beta_{-p,j}^{SCP(n_c)2} + \sum_{j\in D_{pb}}\beta_{-p,j}^{SCP(n_c)2}\right)}_{\text{Without observing}} \underbrace{+ r^2\sigma^2\sum_{j=0}^{n}(1 - g_{p,j})\beta_{-p,j}^{SCP(n_c)2}}_{\text{Observing} D_{pb}}$$
$$= -\left(1 - r\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2 + n_c} - \frac{\sigma^2}{\sigma^2 + n_c + 1}\right)$$
$$- 2\frac{r\sigma^2}{\sigma^2 + n_c + 1}\left(1 - r\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)}\right)\sum_{j\in D_{pb}}\beta_{-p,j}^{SCP(n_c)}$$
$$- r^2\sigma^2\left[\sum_{j\in D_{pb}}\beta_{-p,j}^{SCP(n_c)2} - \frac{1}{\sigma^2 + n_c + 1}\left(\sum_{j\in D_{pb}}\beta_{-p,j}^{SCP(n_c)}\right)^2\right]$$
$$= -\left(1 - r\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2 + n_c} - \frac{\sigma^2}{\sigma^2 + n_c + 1}\right)$$
$$- 2\frac{r\sigma^2}{\sigma^2 + n_c + 1}\left(1 - r\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)}\right)\beta_{-p,c}^{SCP(n_c)}$$
$$- r^2\sigma^2\beta_{-p,c}^{SCP(n_c)2}\left[1 - \frac{1}{\sigma^2 + n_c + 1}\right],$$

and

$$\Delta U_{cf}^{SCP}(n_c) = -\underbrace{\frac{\sigma^2\left(1 - r\sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)} - r\sum_{j\in D_{cf}}\beta_{-c,j}^{SCP(n_c)}\right)^2}{\sigma^2 + n_c + 1}}_{\text{Observing } D_{cf}} + \underbrace{\frac{\sigma^2\left(1 - r\sum_{j=0}^{n} g_{c,j}\beta_{-c,j}^{SCP(n_c)}\right)^2}{\sigma^2 + n_c}}_{\text{Without observing}}$$
$$\underbrace{- r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{c,j})\beta_{-c,j}^{SCP(n_c)2} - \sum_{j\in D_{cf}}\beta_{-c,j}^{SCP(n_c)2}\right)}_{\text{Observing } D_{cf}} \underbrace{+ r^2\sigma^2\left(\sum_{j=0}^{n}(1 - g_{c,j})\beta_{-c,j}^{SCP(n_c)2}\right)}_{\text{Without observing}}$$
$$= \left(1 - r\sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)}\right)^2\left(\frac{\sigma^2}{\sigma^2 + n_c} - \frac{\sigma^2}{\sigma^2 + n_c + 1}\right)$$
$$+ 2\frac{r\sigma^2}{\sigma^2 + n_c + 1}\left(1 - r\sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)}\right)\sum_{j\in D_{cf}}\beta_{-c,j}^{SCP(n_c)}$$

$$+ r^2\sigma^2 \left[\sum_{j\in D_{cf}} \beta_{-c,j}^{SCP(n_c)2} - \frac{1}{\sigma^2+n_c+1} \left(\sum_{j\in D_{cf}} \beta_{-c,j}^{SCP(n_c)} \right)^2 \right]$$

$$= \left(1 - r \sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)} \right)^2 \left(\frac{\sigma^2}{\sigma^2+n_c} - \frac{\sigma^2}{\sigma^2+n_c+1} \right)$$

$$+ 2\frac{r\sigma^2}{\sigma^2+n_c+1} \left(1 - r \sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)} \right) \beta_{-c,p}^{SCP(n_c)}$$

$$+ r^2\sigma^2\beta_{-c,p}^{SCP(n_c)2} \left[1 - \frac{1}{\sigma^2+n_c+1} \right]$$

Hence, to prove that $\Delta U_{cf}^{SCP}(n_c) < -\Delta U_{pb}^{SCP}(n_c)$ it is sufficient to show that:

(i) $\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)} < \sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)}$, and

(ii) $\beta_{-c,p}^{SCP(n_c)} < \beta_{-p,c}^{SCP(n_c)}$.

Notice that:

$$\sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)} = \beta_{-p,0}^{SCP(n_c)} + (n_c-1)\beta_{-p,c}^{SCP(n_c)} = 1 - \beta_{-p,c}^{SCP(n_c)} - (n-n_c-1)\beta_{-p,p}^{SCP(n_c)}$$

$$\sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)} = \beta_{-c,0}^{SCP(n_c)} + n_c\beta_{-c,c}^{SCP(n_c)} = 1 - (n-n_c)\beta_{-c,p}^{SCP(n_c)}$$

Thus,

$$\sum_{j\notin D_{cf}} g_{c,j}\beta_{-c,j}^{SCP(n_c)} - \sum_{j\notin D_{pb}} g_{p,j}\beta_{-p,j}^{SCP(n_c)} = (n-n_c)\underbrace{\left[\beta_{-p,p}^{SCP(n_c)} - \beta_{-c,p}^{SCP(n_c)}\right]}_{=0 \text{ (Lemma A.11)}} + \underbrace{\beta_{-p,c}^{SCP(n_c)} - \beta_{-p,p}^{SCP(n_c)}}_{>0 \text{ (Lemma A.11)}} > 0.$$

Furthermore, from Lemma A.11, we know that $\beta_{-c,p}^{SCP(n_c)} < \beta_{-p,c}^{SCP(n_c)}$, which guarantees the second sufficient conditions mentioned above. □

The proof of the following theorem is organized in two parts. First, we consider a $SCP(n_c)$ network, and show that we can always find a cost function that satisfies both equilibrium conditions. Next, we consider a $CPOD(n_c)$ network and show that we can always satisfy the equilibrium conditions. Intuitively, if to observe n_c signals is free and to observe n_c+1 signals is infinitely costly, then $CPOD(n_c)$ is an equilibrium.

Proposition A.2. *For any core-periphery network, G, there exists a cost function satisfying Assumption 2 such that G is an equilibrium.*

Proof. Let us begin by considering that G is a $SCP(n_c)$.

Start with an empty network, $n_c = 0$. It is an equilibrium if and only if $\Delta U_{cf}^{SCP}(0) \le c_1$. In particular, given that the benefit of observing a signal is bounded, it is sufficiently to choose c_1 large enough, and $c_j = j*c_1$. Consider now a full network, $n_c = n$. It is an equilibrium if c_n is low enough. Indeed, it is an equilibrium if and only if $c_n \le -\Delta U_{pb}^{SCP}(n_c)$. In particular, it is enough to define $c_j = j*\varepsilon$. Given that the benefit of observing a signal is always strictly positive, for some ε small enough the full network will be an equilibrium.

Let us now consider a $SCP(n_c)$, with $0 < n_c < n$. A particular SCP network with $0 < n_c < n$ players in the core, G, is an equilibrium if and only if $\Delta U_{cf}^{SCP}(n_c) \le c_{n_c} \le -\Delta U_{pb}^{SCP}(n_c)$.

By Lemma A.25, we know that for any set of parameters and n_c, $\Delta U_{cf}^{SCP}(n_c) < -\Delta U_{pb}^{SCP}(n_c)$. Thus it is always possible to find c_n in the above interval.

Let us begin by considering that G is a $CPOD(n_c)$. Consider the following marginal cost sequence,

$$c_j = \begin{cases} j * \varepsilon & if \quad j \leq n_c \\ j * \frac{1}{\varepsilon} & if \quad j > n_c. \end{cases}$$

For a ε small enough, the above marginal cost sequence guarantees that $CPOD(n_c)$ is an equilibrium. □

D Leadership

In this section, we prove Propositions 5 and 6.

D.1 Proof of Proposition 5

In the model with heterogeneous r, agents form K without bearing any cost, that is, the cost of forming up to K connection is zero and after that it becomes prohibitively costly to acquire additional connections. Without loss of generality we sort agents by r: $r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_n$. To prove Proposition 5, we assume that equilibrium network is given by:

$$g_{ij} = \begin{cases} 1, \text{ if } j = 1, \ldots, N_c, \text{ or } j = i, \text{ or } i \leq N_c \text{ and } j = N_c \\ 0, \text{ otherwise} \end{cases} .$$

We show in Lemma A.26 that β_i is weakly decreasing in r_i and, in Lemma A.27, we show that $\beta_h > \beta_l$ implies $\beta_{-i,h} > \beta_{-i,l}$ for $i \neq h, l$. This implies that everyone will look to the same set of players and there is a core-periphery equilibrium. Lemmata A.26 and A.27 heavily rely on expression in Equations (DD.5), (DD.11), and (DD.8).

Lemma A.26. *In the model with heterogeneous r_i, if core player have the lowest r_i, then β_i is weakly decreasing in r_i.*

Proof. We show that $\beta_1 \geq \beta_2 \geq \ldots, \beta_n$ given that $r_1 \leq r_2 < \ldots r_n$. For $j = 1, \ldots, K$ in the core of the network

$$n\beta_j = \sum_{i=1}^{n} g_{ij} \left[\frac{1}{\sigma^2 + K + 1} + \tilde{r}_i \beta_j - \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \right] \tag{DD.44}$$

$$= \frac{n}{\sigma^2 + K + 1} + \beta_j \sum_{i=1}^{n} \tilde{r}_i - \sum_{i=1}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1}, \tag{DD.45}$$

which implies that

$$\beta_j = \beta_c \,\forall i, j = 1, \ldots, K \tag{DD.46}$$

For $j = 0$ in the core of the network

$$n\beta_j = \sum_{i=1}^{n} \left[\tilde{g}_{ij} + \tilde{r}_i \beta_j g_{ij} - \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \tilde{g}_{ij} \right]$$

$$= \frac{n\sigma^2}{\sigma^2 + K + 1} + \beta_j \sum_{i=1}^{n} \tilde{r}_i - \sum_{i=1}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{\sigma^2}{\sigma^2 + K + 1},$$

which implies that

$$\beta_0 = \sigma^2 \beta_c \tag{DD.47}$$

For $j = K + 1$:

$$n\beta_j = n\beta_{cp} = \sum_{i=1}^{n} g_{ij} \left[\frac{1}{\sigma^2 + K + 1} + \tilde{r}_i \beta_j - \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \right] \tag{DD.48}$$

$$= \sum_{i=1}^{K+1} g_{ij} \left[\frac{1}{\sigma^2 + K + 1} + \tilde{r}_i \beta_j - \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \right] \tag{DD.49}$$

$$= \frac{K+1}{\sigma^2 + K + 1} + \beta_{cp} \sum_{i=1}^{K+1} \tilde{r}_i - \sum_{i=1}^{K+1} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1}, \tag{DD.50}$$

Combing with Equation (DD.45) by taking the difference:

$$\begin{aligned}
n\beta_c &= \frac{n}{\sigma^2 + K + 1} + \beta_c \sum_{i=1}^{n} \tilde{r}_i - \sum_{i=1}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \\
n\beta_{cp} &= \frac{K+1}{\sigma^2 + K + 1} + \beta_{cp} \sum_{i=1}^{K+1} \tilde{r}_i - \sum_{i=1}^{K+1} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1}, \\
n\left(\beta_c - \beta_{cp}\right) &= \frac{n-K-1}{\sigma^2 + K + 1} + \left(\beta_c - \beta_{cp}\right) \sum_{i=1}^{K+1} \tilde{r}_i + \beta_c \sum_{i=K+2}^{n} \tilde{r}_i - \sum_{i=K+2}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \\
\left[n - \sum_{i=1}^{K+1} \tilde{r}_i \right] \left(\beta_c - \beta_{cp}\right) &= \frac{n-K-1}{\sigma^2 + K + 1} + \beta_c \sum_{i=K+2}^{n} \tilde{r}_i - \sum_{i=K+2}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \qquad \text{(DD.51)} \\
\left(n - \sum_{i=1}^{K+1} \tilde{r}_i \right) \left(\beta_c - \beta_{cp}\right) &\geq \frac{1}{\sigma^2 + K + 1} \left[n - K - 1 - \sum_{i=K+2}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \right] \\
&\geq \frac{1}{\sigma^2 + K + 1} \left[n - K - 1 - \sum_{i=K+2}^{n} \tilde{r}_i \right] \\
&> \frac{1}{\sigma^2 + K + 1} \left[n - K - 1 - \sum_{i=K+2}^{n} 1 \right] = 0
\end{aligned}$$

Therefore:

$$\beta_c > \beta_{cp} \tag{DD.52}$$

For $j \geq K + 2$ in the periphery of the network

$$\begin{aligned}
n\beta_j &= \sum_{i=1}^{n} g_{ij} \left[\frac{1}{\sigma^2 + K + 1} + \tilde{r}_i \beta_j - \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2 + K + 1} \right] \\
&= \frac{1}{\sigma^2 + K + 1} + \beta_j \tilde{r}_j - \tilde{r}_j \left(\sum_{s=0}^{n} \beta_s g_{js} \right) \frac{1}{\sigma^2 + K + 1} \\
&= \frac{1}{\sigma^2 + K + 1} + \beta_j \tilde{r}_j - \tilde{r}_j \left(\beta_j + \sigma^2 \beta_c + K\beta_c \right) \frac{1}{\sigma^2 + K + 1}
\end{aligned}$$

For $j = K+2, \ldots, n$, we have that $\beta_c > \beta_j$. Combing with Equation (DD.45) by taking the difference:

$$\begin{aligned}
n\beta_c &= \frac{n}{\sigma^2+K+1} + \beta_c \sum_{i=1}^{n} \tilde{r}_i - \sum_{i=1}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2+K+1}, \\
n\beta_j &= \frac{1}{\sigma^2+K+1} + \beta_j \tilde{r}_j - \tilde{r}_j \left(\sum_{s=0}^{n} \beta_s g_{js} \right) \frac{1}{\sigma^2+K+1} \\
n\left(\beta_c - \beta_j\right) &= \frac{n-1}{\sigma^2+K+1} + \beta_c \sum_{i=1, i\neq j}^{n} \tilde{r}_i + \tilde{r}_j \left(\beta_c - \beta_j\right) - \sum_{i=1, i\neq j}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2+K+1} \qquad \text{(DD.53)} \\
\left(n - \tilde{r}_j\right)\left(\beta_c - \beta_j\right) &= \frac{n-1}{\sigma^2+K+1} + \beta_c \sum_{i=1, i\neq j}^{n} \tilde{r}_i - \sum_{i=1, i\neq j}^{n} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2+K+1} \\
&\geq \frac{n-1}{\sigma^2+K+1} + \beta_c \sum_{i=1, i\neq j}^{n} \tilde{r}_i - \sum_{i=1, i\neq j}^{n} \tilde{r}_i \frac{1}{\sigma^2+K+1} \\
&\geq \frac{n-1}{\sigma^2+K+1} + \beta_c \sum_{i=1, i\neq j}^{n} \tilde{r}_i - \frac{n-1}{\sigma^2+K+1} > 0
\end{aligned}$$

Thus:

$$\beta_c > \beta_j \forall j = K+2, \ldots, n. \qquad \text{(DD.54)}$$

For $j = K+2, \ldots, n$, we have that $\beta_{cp} > \beta_j$. Combing with Equation (DD.50) by taking the difference:

$$\begin{aligned}
n\beta_{cp} &= \frac{K+1}{\sigma^2+K+1} + \beta_{cp} \sum_{i=1}^{K+1} \tilde{r}_i - \sum_{i=1}^{K+1} \tilde{r}_i \left(\sum_{s=0}^{n} \beta_s g_{is} \right) \frac{1}{\sigma^2+K+1} \\
&= \frac{K+1}{\sigma^2+K+1} + \beta_{cp} \sum_{i=1}^{K+1} \tilde{r}_i - \sum_{i=1}^{K+1} \tilde{r}_i \left(\beta_{cp} + \sigma^2\beta_c + K\beta_c \right) \frac{1}{\sigma^2+K+1} \\
n\beta_j &= \frac{1}{\sigma^2+K+1} + \beta_j \tilde{r}_j - \tilde{r}_j \left(\beta_j + \sigma^2\beta_c + K\beta_c \right) \frac{1}{\sigma^2+K+1} \\
n\left(\beta_{cp} - \beta_j\right) &= \frac{K}{\sigma^2+K+1} + \beta_{cp} \sum_{i=1}^{K+1} \tilde{r}_i - \sum_{i=1}^{K+1} \tilde{r}_i \left(\beta_{cp} + \sigma^2\beta_c + K\beta_c \right) \frac{1}{\sigma^2+K+1} \\
&\quad - \beta_j \tilde{r}_j + \tilde{r}_j \left(\beta_j + \sigma^2\beta_c + K\beta_c \right) \frac{1}{\sigma^2+K+1} \\
&= \frac{K}{\sigma^2+K+1} - \left(\sum_{i=1}^{K+1} \tilde{r}_i \right) \frac{\sigma^2+K}{\sigma^2+K+1} \left(\beta_c - \beta_{cp}\right) + \tilde{r}_j \frac{\sigma^2+K}{\sigma^2+K+1} \left(\beta_c - \beta_j\right) \qquad \text{(DD.55)}
\end{aligned}$$

Using the fact that $\tilde{r}_i$ are sorted, we have that $\tilde{r}_i \leq \tilde{r}_{K+1} \leq \tilde{r}_j$ for all $i = 1, \ldots, K+1$ and $j > K+1$:

$$\begin{aligned}
\left(\sigma^2+K+1\right) n \left(\beta_{cp} - \beta_j\right) &\geq K - (K+1)\, \tilde{r}_{K+1} \left(\sigma^2+K\right)\left(\beta_c - \beta_{cp}\right) + \tilde{r}_{K+1} \left(\sigma^2+K\right)\left(\beta_c - \beta_j\right) \\
&= K - \tilde{r}_{K+1} \left(\sigma^2+K\right) \left[K\left(\beta_c - \beta_j\right) - (K+1)\left(\beta_{cp} - \beta_j\right) \right],
\end{aligned}$$

which implies that

$$\begin{aligned}
\left(\beta_{cp} - \beta_j\right) \left[\left(\sigma^2+K+1\right) n - \tilde{r}_{K+1} \left(\sigma^2+K\right)(K+1) \right] &\geq K - \tilde{r}_{K+1} \left(\sigma^2+K\right) \left[K\left(\beta_c - \beta_j\right) \right] \\
\left(\beta_{cp} - \beta_j\right) \underbrace{\left[n + \left(\sigma^2+K\right)\left(n - \tilde{r}_{K+1}(K+1)\right) \right]}_{>0} &\geq K - \tilde{r}_{K+1} \left(\sigma^2+K\right) \left[K\left(\beta_c - \beta_j\right) \right]
\end{aligned}$$

$$\geq K\left[1-\tilde{r}_{K+1}\left(\sigma^2+K\right)\beta_c\right]$$
$$>0$$

given that $\tilde{r}_{K+1}<1$ and $\left(\sigma^2+K\right)\beta_c<1$. Thus,

$$\beta_{cp}>\beta_j \forall j=K+2,\ldots,n. \tag{DD.56}$$

Also, notice that:

$$\beta_j\geq\beta_s \Leftrightarrow \tilde{r}_j\leq\tilde{r}_s \tag{DD.57}$$

because we can write $\beta_j-\beta_s$ as follows:

$$\begin{aligned}
n\left(\beta_j-\beta_s\right) &= \left(\beta_j\tilde{r}_j-\beta_s\tilde{r}_s\right)\frac{\sigma^2+K}{\sigma^2+K+1}-\left(\tilde{r}_j-\tilde{r}_s\right)\beta_c\frac{\sigma^2+K}{\sigma^2+K+1}\\
&= \left(\beta_j-\beta_s\right)\tilde{r}_j\frac{\sigma^2+K}{\sigma^2+K+1}-\beta_s\left(\tilde{r}_s-\tilde{r}_j\right)\frac{\sigma^2+K}{\sigma^2+K+1}+\left(\tilde{r}_s-\tilde{r}_j\right)\beta_c\frac{\sigma^2+K}{\sigma^2+K+1}\\
\left[n-\tilde{r}_j\frac{\sigma^2+K}{\sigma^2+K+1}\right]\left(\beta_j-\beta_s\right) &= -\beta_s\left(\tilde{r}_s-\tilde{r}_j\right)\frac{\sigma^2+K}{\sigma^2+K+1}+\left(\tilde{r}_s-\tilde{r}_j\right)\beta_c\frac{\sigma^2+K}{\sigma^2+K+1}\\
&= \left(\tilde{r}_s-\tilde{r}_j\right)\frac{\sigma^2+K}{\sigma^2+K+1}\left[\beta_c-\beta_s\right]\geq 0 \Leftrightarrow \tilde{r}_j\leq\tilde{r}_s
\end{aligned}$$

□

Lemma A.27. *In the model with heterogeneous r_i, if $\beta_h\geq\beta_l$, then $\beta_{-i,h}\geq\beta_{-i,l}\forall i\neq h,l$.*

Proof. We can write $\beta_{-i,j}$ as

$$\beta_{-i,j}=\frac{n}{n-1}\beta_j-\frac{1}{n-1}\lambda_{ij}$$

and for $j\geq 1$ we can write λ_{ij} as

$$\lambda_{ij}=g_{ij}\left[\frac{1}{\sigma^2+K+1}+\tilde{r}_i\beta_j-\tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{1}{\sigma^2+K+1}\right].$$

This proof consists of two parts. First, we show that if $\beta_h\geq\beta_l$, then $\beta_{-i,h}\geq\beta_{-i,l}$ for every $i\geq K+2$ and distinct from h and l. Second, we show that result for $i\leq K+1$.

In the first part of the proof, for a player $i\geq K+2$, we have that $\beta_{-i,j}=\beta_{-i,l}$ for every $j,l\leq K$ because $\beta_j=\beta_l$ for every $j,l\leq K$. To show that $\beta_{-i,K}>\beta_{-i,K+1}$, notice that from Equation (DD.51):

$$\begin{aligned}
\left[n-\sum_{i=1}^{K+1}\tilde{r}_i\right]\left(\beta_c-\beta_{cp}\right) &= \frac{n-K-1}{\sigma^2+K+1}+\beta_c\sum_{i=K+2}^{n}\tilde{r}_i-\sum_{i=K+2}^{n}\tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{1}{\sigma^2+K+1}\\
n\left(\beta_c-\beta_{cp}\right) &= \left(\beta_c-\beta_{cp}\right)\sum_{i=1}^{K+1}\tilde{r}_i+\frac{n-K-1}{\sigma^2+K+1}+\beta_c\sum_{i=K+2}^{n}\tilde{r}_i-\sum_{i=K+2}^{n}\tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{1}{\sigma^2+K+1}\\
&= \left(\beta_c-\beta_{cp}\right)\sum_{i=1}^{K+1}\tilde{r}_i+\frac{n-K-1}{\sigma^2+K+1}+\beta_c\sum_{i=K+2}^{n}\tilde{r}_i-\sum_{i=K+2}^{n}\tilde{r}_i\frac{\sigma^2\beta_c+K\beta_c+\beta_i}{\sigma^2+K+1}\\
&= \left(\beta_c-\beta_{cp}\right)\sum_{i=1}^{K+1}\tilde{r}_i+\frac{n-K-1}{\sigma^2+K+1}+\sum_{i=K+2}^{n}\tilde{r}_i\frac{\beta_c-\beta_i}{\sigma^2+K+1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\beta_{-i,K}-\beta_{-i,K+1} &= \frac{n}{n-1}\left(\beta_c-\beta_{cp}\right)-\frac{1}{n-1}\lambda_{i,K}\\
(n-1)\left(\beta_{-i,K}-\beta_{-i,K+1}\right) &= n\left(\beta_c-\beta_{cp}\right)-\left[\frac{1}{\sigma^2+K+1}+\tilde{r}_i\beta_c-\tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{1}{\sigma^2+K+1}\right]\\
&= n\left(\beta_c-\beta_{cp}\right)-\left[\frac{1}{\sigma^2+K+1}+\tilde{r}_i\beta_c-\tilde{r}_i\left(\sigma^2\beta_c+K\beta_c+\beta_i\right)\frac{1}{\sigma^2+K+1}\right]\\
&= n\left(\beta_c-\beta_{cp}\right)-\frac{1}{\sigma^2+K+1}\left[1+\tilde{r}_i\left(\beta_c-\beta_i\right)\right]\\
&= \left(\beta_c-\beta_{cp}\right)\sum_{i=1}^{K+1}\tilde{r}_i+\frac{n-K-1}{\sigma^2+K+1}+\sum_{s=K+2}^{n}\tilde{r}_s\frac{\beta_c-\beta_s}{\sigma^2+K+1}-\frac{1}{\sigma^2+K+1}\left[1+\tilde{r}_i\left(\beta_c-\beta_i\right)\right]\\
&= \left(\beta_c-\beta_{cp}\right)\sum_{i=1}^{K+1}\tilde{r}_i+\frac{n-K-2}{\sigma^2+K+1}+\sum_{s=K+2,s\neq i}^{n}\tilde{r}_s\frac{\beta_c-\beta_i}{\sigma^2+K+1}\\
&> 0.
\end{aligned}$$

We have $\beta_{-i,K+1}-\beta_{-i,l}=\frac{n}{n-1}\left(\beta_{cp}-\beta_l\right)>0$ for distinct $i\geq K+2$ and $l\geq K+2$, by using Lemma A.26. Finally, for distinct $h\geq K+2$ and $l\geq K+2$ different from i, $\beta_{-i,h}-\beta_{-i,l}=\frac{n}{n-1}\left(\beta_h-\beta_l\right)\geq 0$ if, and only if, $\beta_h\geq\beta_l$.

In the second part of this proof, we show that if $\beta_h\geq\beta_l$, then $\beta_{-i,h}\geq\beta_{-i,l}$ for every $i\leq K+1$ as well. For any two distinct $s, j=1,\ldots,K$, we have $\beta_{-i,j}=\beta_{-i,s}$ because $\beta_j=\beta_s=\beta_c$. For $\beta_{-i,K}$ and $\beta_{-i,K+1}$, we have:

$$\begin{aligned}
\beta_{-i,K}-\beta_{-i,K+1} &= \frac{n}{n-1}\left(\beta_c-\beta_{cp}\right)-\frac{1}{n-1}\left(\lambda_{i,K}-\lambda_{i,K+1}\right)\\
&= \frac{n}{n-1}\left(\beta_c-\beta_{cp}\right)-\frac{1}{n-1}\tilde{r}_i\left(\beta_c-\beta_{cp}\right)\\
&= \frac{n-\tilde{r}_i}{n-1}\left(\beta_c-\beta_{cp}\right)>0.
\end{aligned}$$

To show that $\beta_{-i,K+1}>\beta_{-i,K+2}$, notice that from Equation (DD.55) for any $j\geq K+2$:

$$n\left(\beta_{cp}-\beta_j\right)=\frac{K}{\sigma^2+K+1}-\left(\sum_{i=1}^{K+1}\tilde{r}_i\right)\frac{\sigma^2+K}{\sigma^2+K+1}\left(\beta_c-\beta_{cp}\right)+\tilde{r}_j\frac{\sigma^2+K}{\sigma^2+K+1}\left(\beta_c-\beta_j\right),$$

and therefore

$$\begin{aligned}
\beta_{-i,K+1}-\beta_{-i,K+2} &= \frac{n}{n-1}\left(\beta_{cp}-\beta_{K+2}\right)-\frac{1}{n-1}\lambda_{i,K+1}\\
(n-1)\left(\beta_{-i,K+1}-\beta_{-i,K+2}\right) &= n\left(\beta_{cp}-\beta_{K+2}\right)-\left[\frac{1}{\sigma^2+K+1}+\tilde{r}_i\beta_{cp}-\tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{1}{\sigma^2+K+1}\right]\\
&= n\left(\beta_{cp}-\beta_{K+2}\right)-\frac{1}{\sigma^2+K+1}\left[1+\tilde{r}_i\beta_{cp}(\sigma^2+K+1)-\tilde{r}_i\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\right]\\
&= n\left(\beta_{cp}-\beta_{K+2}\right)-\frac{1}{\sigma^2+K+1}\left[1+\tilde{r}_i\beta_{cp}(\sigma^2+K+1)-\tilde{r}_i\left(\sigma^2\beta_c+K\beta_c+\beta_{cp}\right)\right]\\
&= n\left(\beta_{cp}-\beta_{K+2}\right)-\frac{1}{\sigma^2+K+1}\left[1-\tilde{r}_i(\sigma^2+K)\left(\beta_c-\beta_{cp}\right)\right]\\
&= \frac{K}{\sigma^2+K+1}-\left(\sum_{s=1}^{K+1}\tilde{r}_s\right)\frac{\sigma^2+K}{\sigma^2+K+1}\left(\beta_c-\beta_{cp}\right)+\tilde{r}_j\frac{\sigma^2+K}{\sigma^2+K+1}\left(\beta_c-\beta_j\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\sigma^2 + K + 1}\left[1 - \tilde{r}_i(\sigma^2 + K)\left(\beta_c - \beta_{cp}\right)\right] \\
&= \frac{K-1}{\sigma^2 + K + 1} - \left(\sum_{s=1, s\neq i}^{K+1} \tilde{r}_s\right) \frac{\sigma^2 + K}{\sigma^2 + K + 1}\left(\beta_c - \beta_{cp}\right) + \tilde{r}_j \frac{\sigma^2 + K}{\sigma^2 + K + 1}\left(\beta_c - \beta_j\right) \\
&\geq \frac{K-1}{\sigma^2 + K + 1} - K\tilde{r}_{K+1}\frac{\sigma^2 + K}{\sigma^2 + K + 1}\left(\beta_c - \beta_{cp}\right) + \tilde{r}_{K+1}\frac{\sigma^2 + K}{\sigma^2 + K + 1}\left(\beta_c - \beta_j\right) \\
&= \frac{1}{\sigma^2 + K + 1}\left[K - 1 - \tilde{r}_{K+1}\left(\sigma^2 + K\right)\left(K\left(\beta_c - \beta_{cp}\right) - \left(\beta_c - \beta_j\right)\right)\right] \\
&\geq \frac{1}{\sigma^2 + K + 1}\left[K - 1 - \tilde{r}_{K+1}\left(\sigma^2 + K\right)\left(\beta_c - \beta_{cp}\right)(K-1)\right] \\
&\geq \frac{1}{\sigma^2 + K + 1}(K-1)\left[1 - \tilde{r}_{K+1}\right] \\
&> 0.
\end{aligned}
$$

Finally, for $i \leq K+1$ and distinct $h, l \geq K+2$, $\beta_{-i,h} - \beta_{-i,l} = \frac{n}{n-1}(\beta_h - \beta_l) \geq 0$ if, and only if, $\beta_h \geq \beta_l$. □

D.2 Proof of Proposition 6

In the model with heterogeneous σ, agents form K without bearing any cost, that is, the cost of forming up to K connection is zero and after that it becomes prohibitively costly to acquire additional connections. Without loss of generality we sort agents by r: $\sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \ldots \leq \sigma_n$. To prove Proposition 6, we assume that equilibrium network is given by:

$$
g_{ij} = \begin{cases} 1, \text{ if } j = 1, \ldots, N_c, \text{ or } j = i, \text{ or } i \leq N_c \text{ and } j = N_c \\ 0, \text{ otherwise} \end{cases} .
$$

We show in Lemma A.28 that β_i is weakly decreasing in σ_i and, in Lemma A.29, we show that $\beta_h > \beta_l$ implies $\beta_{-i,h} > \beta_{-i,l}$ for $i \neq h, l$. This implies that everyone will look to the same set of players and there is a core-periphery equilibrium.

Lemma A.28. *In the model with heterogeneous σ_i, if core player have the lowest σ_i, then β_i is weakly decreasing in σ_i.*

Proof. For $j = 1, \ldots, K$, we have

$$
\begin{aligned}
n\beta_j &= \sum_{i=1}^{n} g_{ij}\left[\frac{\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} + \tilde{r}\beta_j - \tilde{r}\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\frac{\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}\right] \\
&= \sigma_j^{-2}\sum_{i=1}^{n}\frac{g_{ij}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} + n\tilde{r}\beta_j - \tilde{r}\sigma_j^{-2}\sum_{i=1}^{n}\left(\sum_{s=0}^{n}\beta_s g_{is}\right) g_{ij}\frac{1}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \\
n(1-\tilde{r})\beta_j &= \sigma_j^{-2}\sum_{i=1}^{n}\frac{g_{ij}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}\left[1 - \tilde{r}\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\right] \\
&= \sigma_j^{-2}\left[\sum_{i=1}^{K+1}\frac{1}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}\left[1 - \tilde{r}\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\right] + \sum_{i=K+2}^{n}\frac{1}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}\left[1 - \tilde{r}\left(\sum_{s=0}^{n}\beta_s g_{is}\right)\right]\right] \\
&= \sigma_j^{-2}\left[\sum_{i=1}^{K+1}\frac{1}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1 - \tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right] + \sum_{i=K+2}^{n}\frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K}\sigma_s^{-2}}\left[1 - \tilde{r}\left(\beta_i + \sum_{s=0}^{K}\beta_s\right)\right]\right] \qquad \text{(DD.58)}
\end{aligned}
$$

Therefore, for $j = 1, \dots, K-1$,

$$\frac{\beta_j}{\beta_{j+1}} = \frac{\sigma_j^{-2}}{\sigma_{j+1}^{-2}} \geq 1.$$

For $j = K+1$, we have:

$$\begin{aligned}
n\beta_j &= \sum_{i=1}^{n} g_{ij} \left[\frac{\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} + \tilde{r}\beta_j - \tilde{r}\left(\sum_{s=0}^{n} \beta_s g_{is}\right) \frac{\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \right] \\
&= \sigma_j^{-2} \sum_{i=1}^{n} \frac{g_{ij}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} + (K+1)\tilde{r}\beta_j - \tilde{r}\sigma_j^{-2} \sum_{i=1}^{n} \left(\sum_{s=0}^{n} \beta_s g_{is}\right) g_{ij} \frac{1}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \\
[n - \tilde{r}(K+1)]\beta_j &= \sigma_j^{-2} \sum_{i=1}^{n} \frac{g_{ij}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{n} \beta_s g_{is}\right) \right] \\
&= \sigma_j^{-2} \left[\sum_{i=1}^{K+1} \frac{1}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{n} \beta_s g_{is}\right) \right] \right] \\
&= \sigma_j^{-2} \left[\sum_{i=1}^{K+1} \frac{1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] \right] \qquad \text{(DD.59)}
\end{aligned}$$

By combing the expression for β_K and β_{K+1}, we have:

$$\begin{aligned}
n(1-\tilde{r})\beta_K &= \sigma_K^{-2} \left[\sum_{i=1}^{K+1} \frac{1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] + \sum_{i=K+2}^{n} \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_i + \sum_{s=0}^{K} \beta_s\right) \right] \right] \\
[n - \tilde{r}(K+1)]\beta_{K+1} &= \sigma_{K+1}^{-2} \left[\sum_{i=1}^{K+1} \frac{1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] \right] \\
n(1-\tilde{r})(\beta_K - \beta_{K+1}) &= \tilde{r}[n-K-1]\beta_{K+1} + \left(\sigma_K^{-2} - \sigma_{K+1}^{-2}\right) \sum_{i=1}^{K+1} \frac{1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] \qquad \text{(DD.60)} \\
&\quad + \sigma_K^{-2} \sum_{i=K+2}^{n} \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_i + \sum_{s=0}^{K} \beta_s\right) \right] \\
&> 0.
\end{aligned}$$

For $j \geq K+2$, we have

$$\begin{aligned}
n\beta_j &= \sum_{i=1}^{n} g_{ij} \left[\frac{\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} + \tilde{r}\beta_j - \tilde{r}\left(\sum_{s=0}^{n} \beta_s g_{is}\right) \frac{\sigma_j^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \right] \\
&= \sigma_j^{-2} \sum_{i=1}^{n} \frac{g_{ij}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} + \tilde{r}\beta_j - \tilde{r}\sigma_j^{-2} \sum_{i=1}^{n} \left(\sum_{s=0}^{n} \beta_s g_{is}\right) g_{ij} \frac{1}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \\
(n - \tilde{r})\beta_j &= \sigma_j^{-2} \sum_{i=1}^{n} \frac{g_{ij}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{n} \beta_s g_{is}\right) \right] \\
&= \sigma_j^{-2} \frac{1}{\sum_{s=0}^{n} g_{js}\sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{n} \beta_s g_{js}\right) \right] \\
&= \sigma_j^{-2} \frac{1}{\sigma_j^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_j + \sum_{s=0}^{K} \beta_s\right) \right]
\end{aligned}$$

$$\left(n-\tilde{r}\left(1-\frac{\sigma_j^{-2}}{\sigma_j^{-2}+\sum_{s=0}^K\sigma_s^{-2}}\right)\right)\beta_j = \frac{\sigma_j^{-2}}{\sigma_j^{-2}+\sum_{s=0}^K\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^K\beta_s\right]$$

$$\beta_j = \frac{\frac{\sigma_j^{-2}}{\sigma_j^{-2}+\sum_{s=0}^K\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^K\beta_s\right]}{n-\tilde{r}+\tilde{r}\frac{\sigma_j^{-2}}{\sigma_j^{-2}+\sum_{s=0}^K\sigma_s^{-2}}}, \tag{DD.61}$$

which is strictly increasing in σ_j^{-2}.

To compare β_{K+1} against β_{K+2}, we first start from Equation (DD.62) write β_{K+1} as follows:

$$[n-\tilde{r}(K+1)]\beta_{K+1} = \sigma_{K+1}^{-2}\left[\sum_{i=1}^{K+1}\frac{1}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right]\right]$$

$$\left[n-\tilde{r}(K+1)+\tilde{r}\sum_{i=1}^{K+1}\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\right]\beta_{K+1} = \left[\sum_{i=1}^{K+1}\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K}\beta_s\right)\right]\right]$$

$$\left[n-\tilde{r}(K+1)+\tilde{r}\frac{(K+1)\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\right]\beta_{K+1} = \frac{(K+1)\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K}\beta_s\right)\right]$$

$$\beta_{K+1} = \frac{\frac{(K+1)\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^K\beta_s\right)\right]}{n-\tilde{r}(K+1)+\tilde{r}\frac{(K+1)\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}} \tag{DD.62}$$

Given that $\frac{(K+1)\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}} > \frac{\sigma_j^{-2}}{\sigma_j^{-2}+\sum_{s=0}^K\sigma_s^{-2}}$ and $K>0$, we have that

$$\beta_{K+1} > \beta_{K+2} \tag{DD.63}$$

Based on Equation (DD.61), for $j \geq K+2, \beta_j$ depends positively on σ_j^{-2} and therefore

$$\beta_j \geq \beta_{j+1} \forall j = K+2, \ldots, n-1. \tag{DD.64}$$

□

Lemma A.29. *In the model with heterogeneous σ_i, if $\beta_h \geq \beta_l$, then $\beta_{-i,h} \geq \beta_{-i,l} \forall i \neq h, l$.*

Proof. We can write $\beta_{-i,j}$ as

$$\beta_{-i,j} = \frac{n}{n-1}\beta_j - \frac{1}{n-1}\lambda_{ij}$$

and for $j \geq 1$ we can write λ_{ij} as

$$\lambda_{ij} = g_{ij}\left[\tilde{r}\beta_j + \frac{\sigma_j^{-2}}{\sum_{s=0}^n g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^n \beta_s g_{is}\right)\right].$$

This proof consists of two parts. First, we show that if $\beta_h \geq \beta_l$, then $\beta_{-i,h} \geq \beta_{-i,l}$ for every $i \geq K+2$ and distinct from h and l. Second, we show that result for $i \leq K+1$.

From Equation (DD.58), we can express $\beta_j - \beta_{j+1}$ for $j = 1, \ldots, K-1$ as follows:

$$\beta_j - \beta_{j+1} = \frac{\sigma_j^{-2}-\sigma_{j+1}^{-2}}{n-n\tilde{r}}\left[\sum_{i=1}^{K+1}\frac{1}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right] + \sum_{i=K+2}^{n}\frac{1}{\sigma_i^{-2}+\sum_{s=0}^K\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_i+\sum_{s=0}^K\beta_s\right)\right]\right]$$

$$= \frac{\sigma_j^{-2} - \sigma_{j+1}^{-2}}{n - n\tilde{r}} \left[\frac{K+1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r} \left(\sum_{s=0}^{K+1} \beta_s \right) \right] + \sum_{i=K+2}^{n} \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r} \left(\beta_i + \sum_{s=0}^{K} \beta_s \right) \right] \right]. \tag{DD.65}$$

In the first part of the proof, for a player $i \geq K+2$ and another agent $j \leq K-1$, we have:

$$\begin{aligned}
(n-1)\left(\beta_{-i,j} - \beta_{-i,j+1}\right) &= n\left(\beta_j - \beta_{j+1}\right) - \left(\lambda_{-i,j} - \lambda_{-i,j+1}\right) \\
&= n\left(\beta_j - \beta_{j+1}\right) - \tilde{r}\left(\beta_j - \beta_{j+1}\right) - \frac{\sigma_j^{-2} - \sigma_{j+1}^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \left(1 - \sum_{s=0}^{n} \beta_s g_{is}\right) \\
&= n\left(\beta_j - \beta_{j+1}\right) - \tilde{r}\left(\beta_j - \beta_{j+1}\right) - \frac{\sigma_j^{-2} - \sigma_{j+1}^{-2}}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left(1 - \beta_i - \sum_{s=0}^{K} \beta_s\right) \\
&= \left(\sigma_j^{-2} - \sigma_{j+1}^{-2}\right) \left\{ \frac{(n-1)(n-\tilde{r})}{n - n\tilde{r}} \left[\frac{K+1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] + \sum_{l=K+2}^{n} \frac{1}{\sigma_l^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_l + \sum_{s=0}^{K} \beta_s\right) \right] \right] \right. \\
&\quad \left. - \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left(1 - \beta_i - \sum_{s=0}^{K} \beta_s\right) \right\} \\
&\geq \left(\sigma_j^{-2} - \sigma_{j+1}^{-2}\right) \left\{ \frac{(n-1)(n-\tilde{r})}{n - n\tilde{r}} \left[\frac{K+1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] + \sum_{l=K+2}^{n} \frac{1}{\sigma_l^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_l + \sum_{s=0}^{K} \beta_s\right) \right] \right] \right. \\
&\quad \left. - \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left(1 - \tilde{r}\left(\beta_i + \sum_{s=0}^{K} \beta_s\right)\right) \right\} \\
&= \left(\sigma_j^{-2} - \sigma_{j+1}^{-2}\right) \left\{ \frac{(n-1)(n-\tilde{r})}{n - n\tilde{r}} \left[\frac{K+1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] + \sum_{l=K+2, l\neq i}^{n} \frac{1}{\sigma_l^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_l + \sum_{s=0}^{K} \beta_s\right) \right] \right] \right. \\
&\quad \left. + \left(\frac{n(n-2) + \tilde{r}}{n(1-\tilde{r})}\right) \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left(1 - \tilde{r}\left(\beta_i + \sum_{s=0}^{K} \beta_s\right)\right) \right\} \\
&\geq 0.
\end{aligned}$$

To show that $\beta_{-i,K} > \beta_{-i,K+1}$ for $i \geq K+2$, notice that from Equation (DD.60):

$$\begin{aligned}
n(1-\tilde{r})\left(\beta_K - \beta_{K+1}\right) = \tilde{r}\left[n - K - 1\right]\beta_{K+1} + \left(\sigma_K^{-2} - \sigma_{K+1}^{-2}\right) \frac{K+1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right] \\
+ \sigma_K^{-2} \sum_{i=K+2}^{n} \frac{1}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\beta_i + \sum_{s=0}^{K} \beta_s\right) \right]
\end{aligned} \tag{DD.66}$$

and therefore

$$\begin{aligned}
(n-1)\left(\beta_{-i,K} - \beta_{-i,K+1}\right) &= n\left(\beta_K - \beta_{K+1}\right) - \lambda_{-i,K} \\
&= n\left(\beta_K - \beta_{K+1}\right) - \tilde{r}\beta_K - \frac{\sigma_K^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}} \left(1 - \beta_i - \sum_{s=0}^{K} \beta_s\right) \\
&= (n-\tilde{r})\left(\beta_K - \beta_{K+1}\right) - \tilde{r}\beta_{K+1} - \frac{\sigma_K^{-2}}{\sigma_i^{-2} + \sum_{s=0}^{K} \sigma_s^{-2}} \left(1 - \beta_i - \sum_{s=0}^{K} \beta_s\right) \\
&= \frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})} \tilde{r}\left[n - K - 1\right]\beta_{K+1} + \frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})} \left(\sigma_K^{-2} - \sigma_{K+1}^{-2}\right) \frac{K+1}{\sum_{s=0}^{K+1} \sigma_s^{-2}} \left[1 - \tilde{r}\left(\sum_{s=0}^{K+1} \beta_s\right) \right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\sigma_K^{-2}\sum_{l=K+2}^{n}\frac{1}{\sigma_l^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_l+\sum_{s=0}^{K}\beta_s\right)\right]-\tilde{r}\beta_{K+1}-\frac{\sigma_K^{-2}}{\sigma_i^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left(1-\beta_i-\sum_{s=0}^{K}\beta_s\right)\\
&=\tilde{r}\frac{(n-K-2)(n-1)(n-\tilde{r})+n(n-2)+\tilde{r}}{n(1-\tilde{r})}\beta_{K+1}+\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\left(\sigma_K^{-2}-\sigma_{K+1}^{-2}\right)\frac{K+1}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right]\\
&\quad+\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\sigma_K^{-2}\sum_{l=K+2,l\neq i}^{n}\frac{1}{\sigma_l^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_l+\sum_{s=0}^{K}\beta_s\right)\right]\\
&\quad+\left(\frac{n(n-2)+\tilde{r}}{n(1-\tilde{r})}\right)\frac{\sigma_K^{-2}}{\sigma_i^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left(1-\beta_i-\sum_{s=0}^{K}\beta_s\right)\\
&>0.
\end{aligned}
$$

For $i \geq K+2$ and distinct $h, l \geq K+1$, we have $\beta_{-i,h}-\beta_{-i,l}=\frac{n}{n-1}\left(\beta_h-\beta_l\right)\geq 0$ if, and only if, $\beta_h \geq \beta_l$, which concludes the first part of this proof.

In the second part of this proof, we focus on player $i \leq K+1$. For another agent $j \leq K-1$, we can use Equation (DD.65) to have:

$$
\begin{aligned}
(n-1)\left(\beta_{-i,j}-\beta_{-i,j+1}\right)&=n\left(\beta_j-\beta_{j+1}\right)-\left(\lambda_{-i,j}-\lambda_{-i,j+1}\right)\\
&=n\left(\beta_j-\beta_{j+1}\right)-\tilde{r}\left(\beta_j-\beta_{j+1}\right)-\frac{\sigma_j^{-2}-\sigma_{j+1}^{-2}}{\sum_{s=0}^{n}g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{n}\beta_s g_{is}\right)\\
&=n\left(\beta_j-\beta_{j+1}\right)-\tilde{r}\left(\beta_j-\beta_{j+1}\right)-\frac{\sigma_j^{-2}-\sigma_{j+1}^{-2}}{\sum_{s=0}^{n}g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K+1}\beta_s\right)\\
&=\left(\sigma_j^{-2}-\sigma_{j+1}^{-2}\right)\Bigg\{\frac{(n-1)(n-\tilde{r})}{n-n\tilde{r}}\left[\frac{K+1}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right]+\sum_{l=K+2}^{n}\frac{1}{\sigma_l^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_l+\sum_{s=0}^{K}\beta_s\right)\right]\right]\\
&\qquad-\frac{1}{\sum_{s=0}^{n}g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K+1}\beta_s\right)\Bigg\}\\
&=\left(\sigma_j^{-2}-\sigma_{j+1}^{-2}\right)\Bigg\{\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\left[\frac{K}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right]+\sum_{l=K+2}^{n}\frac{1}{\sigma_l^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_l+\sum_{s=0}^{K}\beta_s\right)\right]\right]\\
&\qquad+\left(\frac{n(n-2)+\tilde{r}}{n(1-\tilde{r})}\right)\frac{1}{\sum_{s=0}^{n}g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K+1}\beta_s\right)\Bigg\}\\
&\geq 0.
\end{aligned}
$$

To show that $\beta_{-i,K}>\beta_{-i,K+1}$, we use Equation (DD.66):

$$
\begin{aligned}
(n-1)\left(\beta_{-i,K}-\beta_{-i,K+1}\right)&=n\left(\beta_K-\beta_{K+1}\right)-\left(\lambda_{-i,K}-\lambda_{-i,K+1}\right)\\
&=n\left(\beta_K-\beta_{K+1}\right)-\tilde{r}\left(\beta_K-\beta_{K+1}\right)-\frac{\sigma_K^{-2}-\sigma_{K+1}^{-2}}{\sum_{s=0}^{n}g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K+1}\beta_s\right)\\
&=\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\tilde{r}\left[n-K-1\right]\beta_{K+1}+\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\left(\sigma_K^{-2}-\sigma_{K+1}^{-2}\right)\frac{K+1}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right]\\
&\quad+\frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\sigma_K^{-2}\sum_{i=K+2}^{n}\frac{1}{\sigma_i^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_i+\sum_{s=0}^{K}\beta_s\right)\right]-\frac{\sigma_K^{-2}-\sigma_{K+1}^{-2}}{\sum_{s=0}^{n}g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K+1}\beta_s\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\tilde{r}\left[n-K-1\right]\beta_{K+1} + \frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\left(\sigma_K^{-2}-\sigma_{K+1}^{-2}\right)\frac{K}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K+1}\beta_s\right)\right] \\
&\quad + \frac{(n-1)(n-\tilde{r})}{n(1-\tilde{r})}\sigma_K^{-2}\sum_{i=K+2}^{n}\frac{1}{\sigma_i^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\left(\beta_i+\sum_{s=0}^{K}\beta_s\right)\right] \\
&\quad + \frac{n(n-2)+\tilde{r}}{n(1-\tilde{r})}\frac{\sigma_K^{-2}-\sigma_{K+1}^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K+1}\beta_s\right) \\
&> 0.
\end{aligned}
$$

To show that $\beta_{-i,K+1} > \beta_{-i,K+2}$, notice that from Equations (DD.61) and (DD.62) for any $j \geq K+2$:

$$
\begin{aligned}
\beta_{K+2} &= \frac{\frac{\sigma_{K+2}^{-2}}{\sigma_{K+2}^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]}{n-\tilde{r}+\tilde{r}\frac{\sigma_j^{-2}}{\sigma_j^{-2}+\sum_{s=0}^{K}\sigma_s^{-2}}} \leq \frac{\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]}{n+\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)} \\
\beta_{K+1} &= \frac{(K+1)\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\left(\sum_{s=0}^{K}\beta_s\right)\right]}{n-\tilde{r}(K+1)\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(n-1)\left(\beta_{-i,K+1}-\beta_{-i,K+2}\right) &= n\left(\beta_{K+1}-\beta_{K+2}\right)-\lambda_{i,K+1} \\
&= n\left(\beta_{K+1}-\beta_{K+2}\right)-\tilde{r}\beta_{K+1}-\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{n} g_{is}\sigma_s^{-2}}\left(1-\sum_{s=0}^{n}\beta_s g_{is}\right) \\
&= n\left(\beta_{K+1}-\beta_{K+2}\right)-\tilde{r}\beta_{K+1}-\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K}\beta_s\right)+\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\beta_{K+1} \\
&\geq n\left(\beta_{K+1}-\frac{\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]}{n+\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}\right)-\tilde{r}\beta_{K+1}-\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left(1-\sum_{s=0}^{K}\beta_s\right)+\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\beta_{K+1} \\
&= \frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]\left\{n\left(\frac{(K+1)}{n+(K+1)\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}-\frac{1}{n+\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}\right)\right. \\
&\qquad \left. -\tilde{r}\frac{(K+1)}{n+(K+1)\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}-1\right\}+\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\beta_{K+1} \\
&= \frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]\left\{\frac{(n-\tilde{r})(K+1)}{n+(K+1)\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}-\frac{n}{n+\tilde{r}\left(\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}-1\right)}-1\right\} \\
&\qquad +\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\beta_{K+1} \\
&\geq \frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]\left\{\frac{(n-\tilde{r})(K+1)}{n-\tilde{r}(K+1)}-\frac{n}{n-\tilde{r}}-1\right\}+\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\beta_{K+1} \\
&= \frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\left[1-\tilde{r}\sum_{s=0}^{K}\beta_s\right]\frac{n(n(K-1)+\tilde{r})}{(n-\tilde{r})(n-r(K+1))}+\frac{\sigma_{K+1}^{-2}}{\sum_{s=0}^{K+1}\sigma_s^{-2}}\beta_{K+1}
\end{aligned}
$$

$> 0.$

Finally, for $i \leq K + 1$ and distinct $h, l \geq K + 2$, $\beta_{-i,h} - \beta_{-i,l} = \frac{n}{n-1} (\beta_h - \beta_l) \geq 0$ if, and only if, $\beta_h \geq \beta_l$. □

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